Peaches, Lemons, and Cookies: Designing Auction Markets with Dispersed Information

Ittai Abraham [*], Susan Athey [†], Moshe Babaioff [‡], Michael Grubb [§]

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Abstract

This paper studies the role of information asymmetries in second price, common value auctions. Motivated by information structures that arise commonly in applications such as online advertising, we seek to understand what types of information asymmetries lead to substantial reductions in revenue for the auctioneer. One application of our results concerns online advertising auctions in the presence of "cookies," which allow individual advertisers to recognize advertising opportunities for users who, for example, are customers of their websites. Cookies create substantial information asymmetries both ex ante and at the interim stage, when advertisers form their beliefs. The paper proceeds by first introducing a new refinement, which we call "tremble robust equilibrium" (TRE), which overcomes the problem of multiplicity of equilibria in many domains of interest. Second, we consider a special information structure, where only one bidder has access to superior information, and show that the seller's revenue in the unique TRE is equal to the expected value of the object conditional on the lowest possible signal, no matter how unlikely it is that this signal is realized. Thus, if cookies identify especially good users, revenue may not be affected much, but if cookies can (even occasionally) be used to identify very poor users, the revenue consequences are severe. In the third part of the paper, we study the case where multiple bidders may be informed, providing additional characterizations of the impact of information structure on revenue. Finally, we consider richer market designs that ensure greater revenue for the auctioneer, for example by auctioning the right to participate in the mechanism.

[*] ittaia@microsoft.com
[†] athey@fas.harvard.edu
[‡] moshe@microsoft.com
[§] mgrubb@mit.edu

1 Introduction

At least since Milgrom and Weber (1982a)'s classic paper, economists have studied the role of information revelation in the design of common value auctions. Milgrom and Weber (1982a)'s linkage principle shows that the auctioneer typically benefits by releasing information publicly to all bidders. In many important classes of applications, however, the information revelation problem is more subtle. The auctioneer may not be able to directly observe and release information, but rather has the option to allow bidders to assess information on their own. The auctioneer may not be able to verify whether and how bidders exercise this option, and the content of the information remains the private information of bidders.

This type of problem arises in the classic examples of common value auctions, auctions for natural resources such as oil and timber: in principle, the auctioneer can either limit or facilitate access to bidders seeking to do seismic surveys or cruise tracts of timber. But there are many other applications as well. In used car auctions, the auctioneer has some control over the type and extent of inspections potential buyers may do. In internet car auctions, some buyers may be local and have the ability to inspect a car in person; the seller can choose whether to allow this or not. In auctions for financial assets, some bidders may have access to better information about the assets. In all of these cases, some bidders may not be able to directly verify whether other bidders have access to superior information in a particular auction.

This paper develops new theoretical results about the impact of the information structure on revenue in common value auctions, focusing on situations where there may be strong asymmetries of information at either the ex ante (before bidders observe their signals) or the interim stage (after observing their signals). The primary motivating application for our study is online advertising. In the US, the fast growing online-advertising market is expected to capture 15% of total US ad spending in 2010 ($25Bn) and grow to more than 40% of ad spending by 2014 ($40Bn) (eMarketer 2010). A large part of the success of search advertising ($12Bn of U.S. online advertising 2010 (Morrison 2010)) is undoubtedly due to advertisers' ability to target advertising to specific audiences by placing ads on keywords that match to search terms entered by users into search engines. Traditionally, display advertising ($9Bn of U.S. online advertising in 2010 (Morrison 2010)) has been less targeted, or targeted based on broad categories of users (e.g. "sports enthusiasts") identified by the publisher who sells the impression, typically based on the user's browsing behavior on that publisher. However, a growing trend is that advertisers are targeting display ads with increasing sophistication by tracking web surfers using cookies (Helft and Vega 2010). Cookies placed on users' computers by specific web sites can be used to match a user with information such as the user's order history with an online retailer, their recent history of airline searches on a travel website, or their browsing and clicking behavior across a network of online publishers (such as publishers on the same advertising network). A critically important difference between the two cases is the source of information. In the case of search advertising or traditional display advertising, the publisher has as much or more information than the advertiser, and the information is disclosed to all ad buyers symmetrically. In cookie-based display-advertising, however, ad buyers bring their own private information collected via cookies stored on web surfers' computers.

Although there are a variety of mechanisms for selling display advertising, auctions are a leading method, especially for "remnant" inventory, and it is in these markets (where in the absence of targeting, impressions may have a fairly low value) that cookies potentially play a very important role. For example, Google's ad exchange is currently described as a second-price auction that takes place in real time: that is, at the moment an internet user views a page on an internet publisher, a call is made to the ad exchange, bidders on the exchange instantaneously view information provided by the exchange about the publisher and the user as well as any cookies they may have for the individual user, and based on that information, place a bid. The cookie is only meaningful to the bidder if it belongs directly to the bidder (e.g., Amazon.com may have a cookie on the machines of regular customers), or if the bidder has purchased access to specific cookies from a third-party information broker. Cookie-based bidding potentially makes display auctions inherently asymmetric at both the ex ante and the interim stage. At the ex ante stage, bidders may vary greatly in their likelihood of holding informative cookies, both because popular websites have more opportunities to track visitors and because different sites vary in the sophistication of their tracking technologies. At the interim stage, for a particular impression, a typical bidder may have only a small chance of having a relevant cookie, but bidders who do, have a substantial information advantage relative to those who do not.

If cookies only provided advertisers private-value information, then increasing sophistication in the prevalence and use of cookies by advertisers would present ad inventory sellers a two-way trade-off between better matching of advertisements with impressions and reduced competition in thinner markets (Levin and Milgrom 2010). In such a private value setting, Board (2009) shows that irrespective of such asymmetry, more cookies and more targeting always increase second-price auction revenue as long as the market is sufficiently thick. However, cookies undoubtedly also contain substantial common value information. (For instance, when one bidder has a cookie which identifies an impression as due to web-bot rather than a human, the impression is of zero value to all bidders.) As a result, the inherent asymmetry created by cookies can lead to cream skimming or lemons-avoidance by informationally advantaged bidders, with potentially dire consequences for seller revenues.

Thus, a designer of online advertising markets (or other markets with similar informational issues) faces an interesting set of market design problems. One question is whether the market should encourage or discourage the use of cookies, and how the performance of the market will be affected by increases in the prevalence of cookies. This is within the control of the market designer: in display advertising, it is up to the marketplace to determine how products are defined. All advertising opportunities from a given publisher can be grouped together, for example. Google's ad exchange reportedly does not support revealing all possible cookies. A second market design question concerns the allocation problem: if an auction is to be used, what format performs best? Both first and second price auctions are used in the industry. There are a number of other design questions, as well, including whether reserve prices, entry fees, or other modifications to a basic auction should be considered.

In order to understand the market design tradeoffs involved in an environment with these kinds of information asymmetries, the first part of our paper develops a model of pure common-value second-price auctions. Perhaps surprisingly, the existing literature leaves a number of questions open. For example, while it is well known that the presence of an informationally-advantaged bidder will substantially reduce seller revenues in a sealed-bid first-price auction (FPA) for an item with common value (Milgrom and Weber 1982b, Engelbrecht-Wiggans, Milgrom and Weber 1983, Hendricks and Porter 1988), substantially less is known about the same issue in the context of second-price auctions. One of the main impediments to progress has been the well known multiplicity of Bayesian Nash Equilibria in second-price common-value auctions (Milgrom 1981). As a consequence, little is known about what types of information structures lead to more or less severe reductions in revenue.

In order to address the multiplicity problem, we begin by suggesting a new refinement, *tremble robust equilibrium*.[1] Tremble robust equilibrium (TRE) selects only Bayesian Nash Equilibria that are near to an equilibrium (in undominated bids) of a perturbed game in which a random bidder enters with vanishingly small probability ε and then bids smoothly over the support of valuations. In addition to capturing an aspect of the real-world uncertainty faced by bidders in the kinds of applications we are interested in, we argue that this refinement has a number of attractive properties. In many cases, this refinement selects a unique equilibrium. In our setting, when bidders are ex ante symmetric it selects the symmetric equilibrium studied by Milgrom and Weber (1982a) in a setting with continuous signals (we have not yet applied our TRE refinement to Milgrom and Weber's (1982a) model). Moreover it rules out intuitively unappealing equilibria in which uninformed bidders bid aggressively because they can rely on others to set fair prices.

We then proceed to analyze a number of special cases of common value second price auctions using the TRE refinement. To develop some intuition about our main results, consider first a very simple example of an information structure in a common value auction. Only one bidder uses cookie tracking (that is, only one bidder is privately informed), and the bidder can only determine the presence or absence of the cookie: that is, informed bidder has a binary signal which either takes on the value {no-cookie} or {cookie}. The other bidders cannot assess the existence of the cookie for a particular impression (though they know the overall information structure, including the probability of cookies). Apart from the restriction to a binary rather than continuous signal, this corresponds to the setting of informational advantage studied by Milgrom and Weber (1982b) in first-price auctions.

We show that for this simple information structure, there is a unique TRE in the second-price auction, one with intuitive appeal. We are then able to address some interesting comparative statics questions about when, and why, different kinds of information asymmetries can have dramatically different impacts on revenue.

Consider two cases within this simple information structure. In the first case, cookies identify "peaches," or high-value impressions. This is perhaps the most natural assumption - someone who has been to an advertiser's website before is more likely to be an active internet shopper than a random web surfer. In the second case, cookies identify "lemons," or low-value impressions. This might occur if a prior visit indicates the surfer is in fact a web-bot and not a real person. In both cases, information is otherwise symmetric across bidders, and the value of the impression is common to all bidders.

At first, it might seem that for both the "lemons" and "peaches" cases, there could be dire consequences for revenue, due to the extreme adverse selection: one bidder has strictly better information than the others. However, the surprising result is that in the "peaches" case, revenue loss is minimal. In contrast, in the "lemons" case, revenue collapses to the value of the "lemons," even if the probability that an impression has a cookie is arbitrarily close to zero. This result contrasts with that of Engelbrecht-Wiggans et al. (1983), which shows that the revenue losses from a first-price auction should be proportional to proportion of impressions that have cookies for both lemons and peaches. Putting our results together with Engelbrecht-Wiggans et al. (1983), it follows that a first-price auction will perform substantially better in an environment where one bidder has access to relatively rare cookies for "lemons."[2]

We next generalize the information structure, allowing for cookies (signals) to be more

[1]In Section 3 we discuss some standard refinements and explain why they do not adequately address the multiplicity problem in common value SPA.

[2]We find that first price auction revenues are always higher than second price auction revenues when only one bidder is informed, but the difference is on the order of ϵ^2 when the informed bidder has access to relatively rare cookies for "peaches", that arrive at rate ϵ.

richly informative, and the informed bidder to have a signal drawn from a finite set. We show that there is a *unique* TRE of the second-price auction. In that TRE the informed bidder is bidding his posterior given his signal, while every uninformed bidder bids the object's expected value conditional on the worst signal being realized (the signal of the informed bidder with lowest posterior), and the revenue equals to this bid. Thus the TRE is one which, like Akerlof's (1970) classic "market for lemons," uninformed buyers are not willing to pay more than their lowest possible value. To understand the role of the refinement, we note that there are Bayesian Nash equilibria with higher revenues, in which a single uninformed bidder bids more aggressively and relies on the informed bidder to bid his expected value and set a "fair" price. Our refinement rules these equilibria out because, in a nearby perturbed auction with a random bidder, aggressive bids by an uninformed bidder would sometimes win at a high price set by the random bidder rather than a fair price set by the informed bidder and hence be unprofitable.

Beyond the initial case of a single informed bidder, we have begun to extend our analysis to the general case of multiple informed bidders with richer signal structures. Our progress so far involves a trade-off with respect to relaxing assumptions, where we have pushed the model in two directions. First, we consider domains in which multiple informed bidders receive discrete signals but restrict attention to information structures that satisfy the *strong-high-signal* property. The strong high signal property is sufficient to ensure the existence of a *unique* TRE, as in the single informed bidder case. Second, we consider all monotonic domains with two bidders, each with a binary-signal (where each signal is either low or high). For any such domain we provide a TRE and prove it is *unique*.

Turning to the case of multiple informed bidders, we begin by defining the strong-high-signal property. We say that an individual bidder's signal realization is *strong* if learning signal realizations from other bidders would not effect the bidder's estimate of the value. We say that a strong signal is *high* if the expected value given that signal is at least as high as the value given any other profile of signals in the domain. The strong-high-signal property is defined recursively. First, it requires that in the domain there exists at least one strong high signal, and second, that the restricted domain that conditions on that signal *not* being realized recursively satisfies the same strong-high-signal property. For domains that satisfy the strong-high-signal property we show that the profile of strategies in which each agent bids the posterior given his signal and the *worst* feasible combination of signals of the others, is indeed the unique TRE.

A domain with only a single informed bidder always satisfies the strong-high-signal property. With a single informed bidder, the informed bidder's high signal is both a strong signal (the informed bidder's posterior belief given the signal is independent of the signals of the others) and a high signal (the expected value given the signal equals the maximum value given any feasible combination of signals for all agents). Moreover, if we condition on that signal *not* being realized and consider the domain with that restriction, that domain also satisfies the condition. (In particular, a domain in which all agents are uninformed satisfies the condition as all signals are strong high signals.) We see that the single informed bidder results are now a special case of the result for domains satisfying strong-high-signal property. The unique TRE that is selected is exactly the one described above: The informed is bidding the posterior given his signal, while every uninformed bidder bids the object's expected value conditional on the worst signal being realized (the signal of the informed bidder with lowest posterior).

Next, we consider a two-bidder binary-signal model, assuming without loss of generality that $Pr[H_1, L_2](1 - v(H_1, L_2)) \leq Pr[L_1, H_2](1 - v(L_1, H_2))$. Here $Pr[H_1, L_2]$ is the probability bidder 1 receives a high signal but bidder 2 receives a low signal and $v(H_1, L_2)$ is the object's value in that event. We show that in the *unique* TRE the following holds. Both bidders bid $v(L_1, L_2) = 0$ conditional on receiving a low signal. Conditional on receiving a high signal, bidder 1 (called the strong bidder) always bids aggressively at the objects maximum possible value $v(H_1, H_2) = 1$. However, conditional on receiving a high signal, bidder 2 (called the weak bidder) mixes between aggressive and defensive bids, bidding 1 with probability $\frac{Pr[H_1,L_2](1-v(H_1,L_2))}{Pr[L_1,H_2](1-v(L_1,H_2))}$ and $v(L_1, H_2)$ with the remaining probability.

Note that bidder 1 is called the strong bidder if $Pr[H_1, L_2](1-v(H_1, L_2)) \leq Pr[L_1, H_2](1-v(L_1, H_2))$. Bidder 1's *strength* is the inverse of $Pr[H_1, L_2](1 - v(H_1, L_2))$. Conditional on receiving a high signal, $Pr[H_1, L_2](1-v(H_1, L_2))$ captures the "downside risk" faced by bidder 1 that bidder 2 may have a low rather than high signal. It takes into account both the likelihood that bidder 2 has a low rather than a high signal ($Pr[H_1, L_2]$) and the size of the value loss ($v(H_1, H_2) - v(H_1, L_2) = 1 - v(H_1, L_2)$) if bidder 2 has a low rather than a high signal. The bidder with the smaller downside risk is called the stronger bidder.

Our analysis of the second-price auction between two informed bidders with cookies encompasses two special cases. The first is that in which one bidder never receives a cookie - or that only one bidder is informed. The second is that in which bidders are symmetric ex ante. These are variations of the polar extremes of ex ante asymmetry and symmetry studied respectively by Milgrom and Weber (1982b) (for first price auctions) and Milgrom and Weber (1982a) (for multiple auction formats). As already stated, the first-price auction has higher revenue under extreme asymmetry when only one bidder is informed. Focusing on the symmetric equilibrium of the SPA when bidders are ex ante symmetric, Milgrom and Weber (1982a) show that the SPA has higher revenue than the FPA. Since, in our setting, the TRE refinement selects the symmetric equilibrium when bidders are symmetric ex ante, the same result applies. (Milgrom and Weber's (1982a) result is proved for continuous signals, but the authors point out in footnote 15 that it is true more generally.) Thus the revenue ranking between first and second price auctions is reversed by sufficient ex ante asymmetry.

Summarizing our findings for second-price, common value auctions, we show that the nature of information asymmetry has strong implications for revenue. When just one bidder is informed, revenue drops to the lowest possible expected value a bidder may have after observing his signal, even if that signal is very rarely realized, implying that revenue may be almost unaffected if cookies identify only peaches, but collapses when they may identify lemons, even if the probability is small. These results with a single informed bidder generalize to settings with multiple informed bidders if the information structure satisfies the strong-high-signal property. When all bidders receive either peaches or lemons information, if all bidders are informed about peaches revenue will be high, while if at least one is informed about lemons revenue will collapse. Nevertheless, when multiple bidders are informed but the strong-high-signal property is relaxed, revenue need not collapse even when cookies identify lemons. This is true, in particular, with two informed bidders who each might receive a cookie that identifies a lemon (so that the value is 0 unless both receive a high signal). In this case, revenues decrease in proportion to the asymmetry in the likelihood that each bidder is the sole bidder to discover a lemon. At one extreme, when bidders are ex ante symmetric, the seller extracts all surplus. At the other extreme, when the information of bidder 1 completely dominates the information of bidder 2 (so that bidder 2 is never the sole bidder to discover a lemon) the seller extracts no surplus. In other words, revenues are particularly low when one bidder faces a much larger adverse selection problem or *winner's curse* than the other.

Taken together, the findings show that common value second price auctions can be vulnerable to low-revenue outcomes when bidders are asymmetric ex ante. Moreover, low-revenue outcomes are associated with particular forms of asymmetry. It is not sufficient for a bidder to be ex ante better informed than another for revenues to suffer substantially. Rather, the key vulnerability of the second price auction is to ex ante asymmetry with respect to information about lemons. In contrast, the distinction between information about lemons and peaches appears unimportant for first price auction revenues (meaning that they are likely to be a good alternative to the second price auction when the likelihood of discovering lemons is asymmetric across bidders).

So far, we have focused mainly on the costs of information asymmetry, while suppressing any benefit. As mentioned above, the cost-benefit tradeoff between private value information (which we have suppressed) and market thinness has already received some attention in the literature, so we do not revisit that here. Instead, in the last part of the paper, we extend the model in a slightly different direction, allowing for the possibility that cookies contain *action-relevant* information. For instance, when placing a display ad on the New York Times website, Zappos can include a picture of the exact running shoes the New York Times reader had previously been looking at on Zappos.com. This means that the information in cookies can directly increase the value of winning to bidders and create a value advantage as well as an informational advantage. Both the presence of private-value and action-relevant information give clear reasons that ad-buyers and sellers alike benefit from incorporating the inherently asymmetric targeting information from cookies into display-ad auctions. Thus, simply banning cookie based bidding is likely not optimal.

In the final section of our model we discuss a seller's optimal mechanism design problem when cookies contain both action-relevant and common value information. We assume a particular correlation structure in signals: that one and only one bidder receives an informative cookie. In this setting, although all bidders are symmetric ex ante with equal likelihood of receiving a cookie, because only one becomes informed, revenue drops to the expected value conditional on the worst possible cookie.

The revenue drop result follows from the analysis in Section 5.1 of auctions with only one informed bidder. This analysis essentially captures the continuation game that arises at the interim stage once bidders have discovered whether or not they are the single informed bidder. Notice that it does not contradict the result in Section 5.3 that ex ante symmetry leads to full revenue extraction, because that result was specific to the information structure assumed in Section 5.3. The important difference here is that at the interim stage at one case (Section 5.1) at most one bidder has a "high" signal that leads to an interim valuation above the items unconditional expected valuation, while in the other case (Section 5.3) more than one bidder might get such a signal.

Because bidders are ex ante symmetric, we show that full revenue can be extracted by charging symmetric entry fees to participate in the SPA. The TRE refinement is crucial for setting entry fees correctly, as it provides a *unique* prediction of bidder profits in the SPA. Note that this approach is ex ante individually rational, but not interim individually rational.

For bidders with binary signals, we also present an interim individually rational, dominant strategy mechanism that extracts $(1 - 1/n)$-fraction of the social welfare as revenue, using a variant of Myerson's optimal auction for the private value setting. The mechanism runs a SPA on all bids above some reserve price r, and if no such bid exists it sells to a random bidder at a given floor price f (pooling). Loss of revenue arises from lowering the reserve price to satisfy the incentive constraint for bidders with high signals.

2 Related Literature

This paper seeks to understand how revenues in a common value second price auction depend on the structure of information held by bidders. A serious challenge to comparing revenues across different information structures is that for any given information structure there are typically many different equilibria with widely different revenues. For instance, consider a setting commonly studied in the literature: There are two bidders 1 and 2 who receive continuously signals s_1 and s_2 which have marginal distributions $F_1(s_1)$ and $F_2(s_2)$ and the value of the object conditional on the signals is $v(s_1, s_2)$. It is well know that there are a continuum of equilibria (Milgrom 1981). In particular, for any increasing function h, the following bidding strategies form an equilibrium (Milgrom (2004) Theorem 5.4.8):

$$b_1(s_1) = v\left(s_1, h^{-1}(s_1)\right), \; b_2(s_2) = v(h(s_2), s_2). \tag{1}$$

The bidding strategies described by equation (1) imply that bidders 1 and 2 make the same bid whenever bidder 2 has signal s_2 and bidder 1 has signal $h(s_2)$. In other words, the function $h(s_2)$ describes the bidder 1 signal s_1 that ties with s_2 in equilibrium. Because h can be any increasing function, Nash equilibrium makes no prediction about which bidder 1 signal ties with s_2 in equilibrium and hence no useful prediction about revenue. Similar multiplicity arises in our setting in which bidder signals are drawn from discrete and finite support. For instance, in Section 5.1 where we consider the case of a single informed bidder and $n-1$ uninformed bidders, the following is an equilibrium: The informed bidder bids her expected value conditional on her signal, $n-2$ uninformed bidders bid to tie the informed bidder's lowest bid, and the last uninformed bidder bids a bid b that is weakly higher. This is an equilibrium for any such bid b. Thus for any particular signal of the informed bidder, there exists an equilibrium in which the uninformed bidder bids the same amount. Thus revenue could be anywhere between the informed bidder's lowest possible valuation and the full surplus. Similarly, in Section 5.3 where 2 bidders each receive binary signals, it is an equilibrium for the informationally advantaged bidder to always bid 1 when her signal is high and zero otherwise, while the other bidder bids zero following a low signal and mixes between bids of 0 and 1 following a high signal. This is an equilibrium for any mixing probability by the weak bidder. Thus in equilibrium the weak bidder with a high signal can tie with the strong bidder with a low signal, the strong bidder with a high signal, or a mixture of the two. In each case our TRE refinement resolves this multiplicity by selecting a unique equilibrium.

A common approach in the literature with symmetric bidders is to focus on the symmetric equilibrium: $b_i(s_i) = v(s_i, s_i)$ (Milgrom and Weber 1982a, Matthews 1984). As shown by Milgrom and Weber (1982a) and Matthews (1984) others, this selects the equilibrium in which each bidder bids the object's expected value conditional on the highest signal of competing bidders being equal to her own. This excludes extreme equilibria such as one in which one bidder bids an object's maximum value and all other bidders bid zero. Unfortunately it is not clear how the symmetry refinement can be extended to asymmetric environments of the type we are interested in, or why symmetry should be expected in equilibrium.[3] In fact, Klemperer (1998) argues that with *almost common values* all reasonable equilibria are extremely asymmetric.

Recent work by Parreiras (2006), Cheng and Tan (2007), and Larson (2009) introduce perturbations to select a unique equilibrium in two-bidder auctions with continuously distributed

[3] Hausch (1987) selects the $b_i(s_i) = v(s_i, s_i)$ equilibrium in a setting in which asymmetry implies $v(s_i, s_j) \neq v(s_j, s_i)$ for $s_i \neq s_j$. The motivation for this choice is unclear.

signals. Parreiras (2006) perturbs the auction format by assuming that winning bidders pay their own bid rather than the second highest with probability ε, and taking the limit as ε goes to zero. Cheng and Tan (2007) and Larson (2009) introduce private value perturbations to the common value environment and take the limit as these perturbations go to zero. Cheng and Tan (2007) assume private value perturbations are perfectly correlated with common value signals and are symmetric across bidders. The symmetry of perturbations selects a unique equilibrium. Larson (2009) allows for asymmetric perturbations which are assumed to be independent of common value signals and shows that the equilibrium selected depends on the ratio of the standard deviations of the two bidders' private value perturbations. Larson (2009) shows that a weakness of Cheng and Tan's (2007) approach is that it is the assumption of symmetry in perturbations that drives the equilibrium selection. The different choices of perturbations by different authors lead to very different equilibrium selection and conclusions. For instance, Parreiras (2006) show that second-price auctions generate at least as much revenue as first-price auctions even when bidders are asymmetric (given affiliated signals). In contrast, Cheng and Tan (2007) find that first-price auction revenues are strictly higher than second-price auction revenues given any bidder asymmetry ex ante (with independent signals and a submodular value function).

An alternative approach taken in the literature that has been applied to auctions with more than two bidders is to select equilibria that survive iterated deletion of dominated strategies. Harstad and Levin (1985) consider the case in which the first order-statistic of bidders' signals is a sufficient statistic for the object's value in the Milgrom and Weber (1982a) setting with symmetric bidders and continuously distributed signals. For this case, Harstad and Levin (1985) shows that iterated deletion of dominated strategies uniquely selects the symmetric Milgrom and Weber (1982a) equilibrium. Einy, Haimanko, Orzach and Sela (2002) consider the case of asymmetric bidders and discrete signals with finite support. Einy et al. (2002) show that if the information structure is *connected* then iterated deletion of dominated strategies selects a set of equilibria with a unique Pareto-dominant (from bidders' perspective) equilibrium. Malueg and Orzach (2009) apply Einy et al.'s (2002) refinement in two examples and Malueg and Orzach (2011) apply Einy et al.'s (2002) refinement to the special case of two-bidder auctions with *connected* and *overlapping* information partitions. For a particular one-parameter family of common-value distributions, Malueg and Orzach (2011) find that distributions with sufficiently thin left tails yield lower revenues in second-price auctions than in first-price auctions. The primary drawback to Einy et al.'s (2002) approach is that the required assumptions on the information structure are very restrictive. For instance, we show in Appendix C.2 that Einy et al.'s (2002) connectedness property is strictly more restrictive than our *strong-high-signal* property. Thus connectedness rules out many interesting settings such as our model of two bidders with binary signals in which neither bidder is perfectly informed. Even when one bidder is perfectly informed, iterated deletion of dominated strategies is unhelpful on its own: Any bid by the uninformed bidder between the informed bidder's low (0) and high (1) interim valuations survives, so revenue may be anywhere between 0 and the object's expected value.

An alternative literature on *almost-common-value* auctions perturbs the common value framework by assuming that one bidder has a small value-advantage and is known ex ante to value the object slightly more than other bidders. The common wisdom from early papers which modeled two-bidder auctions is that a slight value advantage causes: (1) the strong bidder to win almost all the time, (2) for revenues to collapse in second-price auctions, and (3) for first-price auctions to generate higher revenue (Bikhchandani 1988, Avery and Kagel 1997, Klemperer 1998, Bulow, Huang and Klemperer 1999). However, more recently Levin and Kagel (2005) show that dramatic revenue losses from small asymmetries rely on the two-bidder assumption and that revenue losses are proportional to the value advantage when there are three or more bidders.

Our approach focuses on the pure common value model where no bidder has a value advantage. We do not wish to restrict the information structure to be symmetric or connected, so cannot focus on symmetric equilibria as in Milgrom and Weber (1982a) or use iterated deletion of dominated strategies as in Einy et al. (2002). Instead we introduce a new refinement, TRE, which selects equilibria near those of a perturbed game with an additional random bidder. This is similar in spirit to Parreiras (2006), Cheng and Tan (2007), and Larson (2009). Unlike Cheng and Tan's (2007) and Larson's (2009) private value perturbation refinements, TRE typically does not need further refinement to select among perturbations since there is often a unique TRE. For instance, the TRE refinement selects the symmetric equilibrium when bidders are ex ante symmetric (providing an additional justification for focusing on such symmetric equilibria). Cheng and Tan's (2007) and Larson's (2009) private value perturbation refinements do the same with the additional assumption that the perturbations be symmetric across bidders. Our finding that sufficient ex ante asymmetry favors first price auctions over second price auctions (reversing Milgrom and Weber's (1982a) result from the symmetric case) is similar to Cheng and Tan's (2007) result that ex ante asymmetry favors first-price auctions but contrasts with Parreiras's (2006) finding that Milgrom and Weber's (1982a) first and second-price auction revenue ranking result is robust to asymmetry. Our finding that revenue losses in second price auctions due to informationally advantaged bidders are much larger when the information advantage concerns "lemons" rather than "peaches" mirrors the cost of private seller information in Akerlof's (1970) market for lemons. However the result is novel as it depends on the tremble robust equilibrium refinement - alternative equilibrium selection rules would lead to a different result.

In the context of analyzing the generalized second price (GSP) auction for sponsored search with independent valuations and complete information, Hashimoto (2010) proposes to refine the set of equilibria by adding a non-strategic random bidder that participates in the auction with small probability. Edelman, Ostrovsky and Schwarz (2007) and Varian (2007) have shown that GSP has an envy-free efficient equilibrium, the main result of Hashimoto (2010) is that this equilibrium does not survive the refinement.

3 The Solution Concept

Consider the following simple scenario. We run a second price auction (with random tie breaking) for a common value good. Assume that the good has only two possible values, P (Peach) and L (Lemon) and it holds that $P > L = 0$. Each value is realized with probability 1/2. There are two agents, one is perfectly informed about the value of the good, while the other only knows the prior. Negative bids are dominated and will never be submitted. What bidding strategies and revenues should we expect?

Nash equilibrium provides no prediction about revenue beyond an upper bound of the full surplus $(L + P)/2$. It is an equilibrium for the informed bidder to bid his value and the uninformed bidder to bid P, which results in full surplus extraction. However, it is also an equilibrium for the uninformed bidder to bid $10P$ and the informed bidder to bid 0, earning 0 revenue.

A natural refinement is to restrict attention to Nash equilibria in which bidders only use undominated strategies. Notice that unlike in the private value model, agents do *not* necessarily have a dominant strategy in a common value second price auction. Indeed, in the scenario described above the informed agent has a dominant strategy (to bid the value given his signal), while the uninformed agent does not. To see that, observe that for any two bids b_1 and b_2 such that $P \geq b_1 > b_2 \geq L$ there exist two strategies of the informed agent such that for one strategy the utility from b_1 is higher, while for the other strategy the utility from b_2 is higher. Bidding b_1 is superior to bidding b_2 when the informed is bidding $(b_1 + b_2)/2$ when the value is P, and bidding L when the value is L. On the other hand bidding b_2 is superior to bidding b_1 when the informed is bidding $(b_1 + b_2)/2$ when the value is L, and bidding L when the value is P (handing out the good items to the other bidder).

Thus ruling out dominated strategies restricts the informed bidder to use her dominant strategy and bid her value. However the only restriction placed on the uninformed bidder is that he not bid less than L or more than P. Revenue could be anywhere between L and the full surplus.

A common approach to the multiplicity problem in the literature is to focus on settings in which bidders are ex ante symmetric and assume bidders bid symmetrically (e.g. Milgrom and Weber (1982a)). Unfortunately this is not applicable when we are studying situations in which bidders are known to be substantially different ex ante. Einy et al. (2002) restrict attention to *sophisticated equilibria* which survive iterative simultaneous maximal elimination of weakly dominated strategies. However, this refinement by itself does not identify a unique outcome. In the current example, for instance, the set of *sophisticated equilibria* are the same as the set of Nash equilibria in undominated strategies. Thus, Einy et al. (2002) further refine the set of sophisticated equilibria by focusing on strategies that *guarantee* a payoff of zero, which does identify a unique outcome in the restricted information structure ("connected domains") they study. Einy et al. (2002) also show that in connected domains uniqueness can be achieved by introducing an additional rational uninformed bidder.[4] Unfortunately, focusing on strategies that guarantee a payoff of zero or introducing another rational uninformed bidder are *not* sufficient to derive a unique outcome even in simple domains such as the one we analyze in which two informed bidders each receive a binary signal.

We also believe that for the example in discussion the natural outcome is that the informed bidder bid her posterior value and the uninformed bidder bid L. As observed by Einy et al. (2002), introducing another rational uninformed bidder provides the needed refinement for some domains. Yet, as this refinement does not provide unique outcome even in some rather simple domains we are interested in, we suggest a different approach that achieves the same outcome for the example in discussion. In our refinement, with some small probability another uninformed bidder enters the auction and bids somewhere between L and P. That "random" bidder is not assumed to act rationally and his sole propose is to make the game "noisy" in order to remove unreasonable equilibria. Indeed, in the example discussed, in the presence of such a bidder if the uninformed bidders bids higher than L she risks overpaying for a low value item without ever winning the high value item at a discount.

We formalize this intuition by considering a perturbation of the game in which with some small probability $\epsilon > 0$ there is an additional bidder that comes to the auction and bids a random value drawn from some distribution which is "nice" (satisfying some simple assumptions: support on the relevant values, differentiability and density that is continuous and positive on the interval). We want to consider only Nash equilibria that are nearby to Nash equilibria (in undominated bids) of such perturbed games.

[4]For the current example this works because in the second stage of iterated elimination, each uninformed agent can never have positive utility by any bid b larger than L, while it can have negative utility if any of the other uninformed agents is bidding more than L but less than b. Thus bidding L is the only strategy that is not eliminated in the second stage.

Returning to the example with one informed bidder and one uninformed bidder, recall that bidding her interim value was the only undominated bid for the informed bidder. Given that the informed bidder bids her posterior value, the presence of a random bidder means that L is the only undominated bid for an uninformed bidder. The informed bidder ensures that the uninformed bidder can never win the object at a discount below value. However the random bidder ensures that any bid above L risks overpaying for a low value object when the random bidder sets the price. Thus bidding above L leads to a strictly negative payoff. We observe that by adding noise a *unique* strategy profile and revenue is predicted.

Motivated by the above we suggest the following refinement of (mixed) Nash Equilibrium for an auction scenario. The solution concept picks a (mixed) Nash equilibrium that is the limit, as ϵ goes to zero, of a series of mixed Nash equilibrium (with support on undominated bids) of each modification of the original game in which another "random" bidder is added with small probability ϵ. The random bidder is bidding a random value drawn from some distribution with support over the "relevant" values, is differentiable and has density that is continuous and positive on the interval. We call such a profile of strategies a *Tremble Robust Equilibrium (TRE)*. The formal definition of this new refinement is presented in Section 4.2. Moreover, if there is a profile of strategies that is (mixed) Nash Equilibrium in *any* such small perturbation of the original game, we call it a *strong Tremble Robust Equilibrium*.

Our TRE refinement similar in spirit to other perturbation based refinements discussed in Section 2 (Parreiras 2006, Cheng and Tan 2007, Larson 2009). One can naturally ask whether instead of using the new refinement of TRE one can use the classical refinement of Tremble Hand Perfect Equilibrium (PE) by Selten (1975). It turns out that the PE solution concept (adjusted to games with infinite sets of actions and incomplete information) is too permissive and does *not* provide the natural unique prediction one would expect in the most basic setting with two agents discussed above: the setting with one informed agent with a binary signal, and one uninformed agent. In Appendix A we show that two extensions by Simon and Stinchcombe (1995) of PE to infinite games (which we adjust to incomplete information) do not provide unique prediction in the above setting. On the other hand, in the same setting, if we restrict the perturbation of the informed agent to be *independent* of his signal then in the unique equilibrium the uninformed is bidding the unconditional expected value of the item, contrary to our expectation.

4 Auctions where each Agent has Finitely Many Signals

We start by presenting out model followed by the refinement.

4.1 The Model

An auctioneer is offering an indivisible good to a set N of n potential buyers. Let Ω be the set of states of the world (possibly infinite). There is a prior distribution $H \in \Delta(\Omega)$ over the states, this prior is commonly known.

Let $\omega \in \Omega$ be the realized state of the world, the state is not observed by the buyers. Each buyer i gets a signal about the state of the world ω from a finite set of signals S_i. We denote $S = S_1 \times S_2 \times \ldots \times S_n$. For every state $\omega \in \Omega$ and buyer i there is a known distribution over signals $d_i(\omega) \in \Delta(S_i)$ which is commonly known; and buyer i gets a private signal $s_i \in S_i$ sampled from $d_i(\omega)$. Signal $s_i \in S_i$ for agent i is *feasible* if agent i receives signal s_i with positive probability, and the vector of signals $s = (s_1, s_2, \ldots, s_n) \in S$ is feasible if it is realized with positive probability. We assume that for every i, every signal $s_i \in S_i$ is feasible.[5] The value of the item to agent i when the state of the world is ω is $v_i(\omega) \geq 0$.

When buyer i's signal s_i was realized to s'_i he updates his belief about his expected value of the good, his posterior belief is that the expected value is $v_i(s'_i) = E[v_i(\omega)|s_i = s'_i]$, where the expectation is taken both over the randomness H that generated ω and the randomness $d_i(\omega)$ that generated the realized signal s'_i. Similarly, we denote by $v_i(s')$ the posterior expected value given that each agent i receives signal s'_i and s' is the vector of received signals, that is $v_i(s') = E[v_i(\omega)|s_i = s'_i \; \forall \; i]$.[6]

Definition 1. *A domain is a* monotonic domain *if for each agent j there exists a linear order over his set of signals S_j, and for every agent i and two feasible vectors of signals s and s' such that $s \leq s'$ (that is, for every j it holds that $s_j \leq s'_j$ according to the linear order on S_j) it holds that $v_i(s) \leq v_i(s')$.*

Let $T_j \subseteq S_j$ be a set of signals for buyer j, and let $T = (T_1, T_2, ..., T_n)$ be a vector of such subsets, one for each buyer. We say that T is feasible if some $t \in T$ is feasible (t is realized with positive probability). For T that is feasible let $v_i(T)$ be the expected value that agent i has for the good, conditional on the signal s_j of each buyer j being from T_j.

A pure strategy μ_i for agent i is a mapping from his signal to his bid: $\mu_i : S_i \rightarrow \Re_+$, that is $\mu_i(s_i) \in \Re_+$. A mixed strategy μ_i for agent i is a mapping from his signal to a distribution over non-negative bids.

4.2 The Refinement

We present the following refinement with the goal of pointing out a unique outcome of the game defined by an auction (specifically we use it for the Second Price Auction (SPA)) in our model. The refinement is defined for every game induced by an auction.

The refinement is based on a random bidder that bids according to a distribution that satisfy some properties.

Let $v_{min} = \min_{s \in S}\{v_i(s)\}$ and $v_{max} = \max_{s \in S}\{v_i(s)\}$ be the minimal and maximal possible value conditional on any signal profile, respectively. Similarly, for a fixed signal $s_i \in S_i$, let $v_{min}(s_i) = \min_{s_{-i} \in \{S_{-i}} v_i(s_i, s_{-i})\}$ and $v_{max} = \max_{s_{-i} \in S_{-i}}\{v_i(s_i, s_{-i})\}$.

Definition 2. *We say that a distribution R is* standard *if the support of R is $[v_{min}, v_{max}]$ (the "relevant" values), R is continuous, strictly increasing and differentiable, and its density r is continuous and positive on the interval.*

Consider an auction and the game λ that is induced by the auction. We next define the game with the random bidder added to it.

Definition 3. *For a standard distribution R and $\epsilon > 0$ define $\lambda(\epsilon, R)$ to be the game induced by λ with the following modification: with probability ϵ there is an additional bidder submitting a bid b sampled according to R. We call $\lambda(\epsilon, R)$ an (ϵ, R)-*tremble of the game λ.

Alternatively, one can think of the (ϵ, R)-tremble of the game λ as a game with $n+1$ agents, the n original agents and a random bidder, that random bidder bids 0 with probability $1 - \epsilon$

[5]This is without loss of generality as we can define S_i to be the set of signal with positive probability of being sampled.

[6]If the vector s' can never be realized we define $v_i(s') = 0$.

and bid according to R with probability ϵ. The unconditional distribution according to which the random bidder is bidding is denoted by $\hat{R}$ and is defined as $\hat{R}(x) = 1 - \epsilon + \epsilon \cdot R(x)$. The density of $\hat{R}(x)$ for every $x > 0$ is $\hat{r}(x) = \epsilon \cdot r(x)$.

Let μ_i be a strategy of agent i. A strategy maps the signal of the agent to distribution over bids. The strategy is a pure strategy if for every signal the mapping is to a single bid. Let μ be a vector of strategies, one for each agent.

Definition 4. *(i) A (pure or mixed) Nash equilibrium μ is a* Tremble Robust Equilibrium (TRE) *of the game λ, if there exists a standard distribution R and a sequence of positive numbers $\{\epsilon_j\}_{j=1}^{\infty}$ such that*

1. *$lim_{j \to \infty} \epsilon_j = 0$.*
2. *μ^{ϵ_j} is a (pure or mixed) Nash equilibrium of the game $\lambda(\epsilon_j, R)$, the (ϵ_j, R)-tremble of the game λ, for every ϵ_j.*
3. *for every agent $i \in N$ and signal $s_i \in S_i$,*
   - *The support of $\mu_i^{\epsilon_j}(s_i)$ is contained in $[v_{min}(s_i), v_{max}(s_i)]$ for every ϵ_j (bidders do not submit dominated bids).*
   - *$\{\mu_i^{\epsilon_j}(s_i)\}_{j=1}^{\infty}$ converges in distribution to $\mu_i(s_i)$.*

(ii) μ is a strong Tremble Robust Equilibrium *if it is a TRE and, in addition, for the decreasing sequence $\{\epsilon_j\}_{j=1}^{\infty}$ satisfying (1) and (2) above, there exists k such that for every $j > k$ in (2) it holds that $\mu^{\epsilon_j} = \mu$.*

5 Common Value SPA Auction

In this section we consider the restriction of the above model to the common value case and study the SPA. When we talk about the *Second Price Auction (SPA) game* we refer to the game induced by a Second Price Auction (SPA) with random tie breaking rule. In the common value model the state of the world determines the quality of the good, and thus determines its value. Thus, in the common value model, there exists a value function v such that when the state of the world is $\omega \in \Omega$, the value of the good is $v(\omega)$, that is, $v_i(\omega) = v(\omega)$ for every bidder i.

5.1 Only One Informed Bidder

We first describe the important special case that only one agent has some information about the state, while all others are completely uninformed (each always gets the same signal). Now, all agents but agent 1 are completely uniformed about the state of the world. We call buyer 1 the *informed buyer* and the rest of the buyers are called the *uninformed buyers*. We denote the set of signals of the informed buyer by S.[7] When the informed buyer's signal s was realized to $s' \in S$, he updates his belief about his expected value of the good, his posterior belief is that the expected value is $E[v(\omega)|s = s']$, where the expectation is taken both over the randomness H that generated ω and the randomness $d(\omega)$ that generated the realized signal s'. We are interested in predicting the equilibrium and the revenue of the second price auction.

The next theorem presents a strong TRE in pure strategies and shows that it is the unique TRE.

[7] This is a slight abuse of notation which we take as the set of signals for every other buyer is a singleton.

Theorem 5. *In any domain with one informed buyer and any number of uninformed buyers, the unique TRE of the SPA game is the profile of strategies μ in which:*

1. *the informed buyer with set of signals S and signal s realized to $s' \in S$, bids $b_I(s') = E[v(\omega)|s = s']$.*
2. *each of the uniformed buyers bids $b_U = \min_{\hat{s} \in S} E[v(\omega)|s = \hat{s}]$.*

Moreover, this profile is a strong TRE in pure strategies.

Proof. To show that μ is a strong TRE of the SPA game it is sufficient to show that it is a pure NE in any (ϵ, R)-tremble of the game. This is indeed true as the strategy of the informed bidder is dominant, thus is a best response to any strategies of the uninformed bidders. Additionally, the strategy of any uninformed bidder is a best response to the dominant strategy played by the informed bidder and the strategies of the other uninformed bidders (it gives 0 utility and no strategy give positive utility). Finally, μ is trivially a pure strategy profile.

Next we show that it is the unique TRE. Clearly the strategy of the informed bidder is the unique strategy in undominated bids (even among mixed strategies) as for any signal his bid is the unique bid that dominates any other bid. For any uninformed bidder, bidding below b_U is dominated by bidding b_U, while bidding above b_U cannot be a best response to the unique strategy of the informed bidder in any (ϵ, R)-tremble of the game (thus will not be a NE in any (ϵ, R)-tremble of the game). □

We stress that the strategy of the uniformed is independent of the probability of the informed buyer receiving the signal that generates the lowest expectation: even a tiny (but positive) probability of receiving a signal is sufficient to cause the uniformed buyers to bid so low.

The following corollary is immediate from Theorem 5, it shows that the revenue of the SPA with only one informed bidder in the unique TRE is as low as it can get in undominated bids.

Corollary 6. *In the unique TRE of the SPA game with one informed buyer and any number of uninformed buyers, the expected revenue of the auctioneer is $R = \min_{\hat{s} \in S} E[v(\omega)|s = \hat{s}]$.*

We point out the connection to the Lemon Market problem (Akerlof 1970): similar Adverse Selection phenomena derive both results. Yet, we note that in the SPA with common value, Adverse Selection by itself does *not* necessarily imply significant drop in revenue in every Nash Equilibrium: it is a Nash Equilibrium for the informed agent to bid according to his signal while the uninformed agent bids *any* value X (as any bid results with 0 utility to the uninformed agent). In particular, the uninformed agent *is able* to win high quality items in NE (unlike in the Market for Lemons). Thus, multiplicity of Nash equilibria as well as the ability of the uninformed party to win high quality items in NE make the common value SPA somewhat different than the Markets for Lemons. Our TRE refinement enables a result in the spirit of Markets for Lemons, by predicting a *unique* NE for which there is indeed drop in revenue.

We next discuss the implications of these results to the revenue of the seller in display advertisement common-value SPA with asymmetric information.

Example 7. *Impressions in display ads auction have various qualities (values) dependent on the likelihood of the user to be influenced by the ad to buy some product. Assume that there are two qualities (common value for an impression), low (L for Lemon) and high (P for Peach), that is $\Omega = \{L, P\}$. A peaches is more valuable than a lemon, that is $v(P) > v(L)$. The commonly known prior is that with probability $p \in (0, 1)$ the impression is a peach P, and* *with probability $1-p$ it is a lemon. Fix any small $\hat{\epsilon} > 0$. We consider the follow two possible information structures.*

*In the case that the informed buyer is $\hat{\epsilon}$-*informed about peaches *the set of signals for the informed is $S = \{\emptyset, S_P\}$. Conditional on the quality being high ($\omega = P$) the informed buyer gets signal S_P with probability $\hat{\epsilon}$, otherwise he gets the signal $\emptyset$.*

*In the case that the informed buyer is $\hat{\epsilon}$-*informed about lemons *the set of signals for the informed is $S = \{\emptyset, S_L\}$. Conditional on the quality being low ($\omega = L$) the informed buyer gets signal S_L with probability $\hat{\epsilon}$, otherwise he gets the signal $\emptyset$.*

Let us now examine the revenue for the two cases. Although in both cases the informed buyer has a tiny probability ($\hat{\epsilon}$) of knowing the exact quality, there is a substantial difference in the revenue the seller gets in the SPA in the unique TRE as of Theorem 5.

When the informed buyer is $\hat{\epsilon}$-informed about peaches the revenue is

$$\min_{\hat{s} \in S} E[v(\omega)|s = \hat{s}] = \min\{E[v(\omega)|s = \emptyset], E[v(\omega)|s = S_P]\}$$

Now, $E[v(\omega)|s = S_P] = v(P)$. To compute $E[v(\omega)|s = \emptyset]$ we observe that

$$Pr[s = \emptyset] = Pr[s = \emptyset|\omega = L]Pr[\omega = L] + Pr[s = \emptyset|\omega = P]Pr[\omega = P]$$

$$= (1-p) + (1-\hat{\epsilon})p = 1 - \hat{\epsilon}p$$

We now use Bayes rule to compute $E[v(\omega)|s = \emptyset]$.

$$E[v(\omega)|s = \emptyset] = \sum_{\omega \in \Omega} v(\omega) Pr[\omega|s = \emptyset] = \sum_{\omega \in \Omega} v(\omega) \frac{Pr[s = \emptyset|\omega]Pr[\omega]}{Pr[s = \emptyset]}$$

$$= v(L)\frac{1-p}{1-\hat{\epsilon}p} + v(P)\frac{(1-\hat{\epsilon})p}{1-\hat{\epsilon}p} = \frac{v(L)(1-p) + v(P)(1-\hat{\epsilon})p}{1-\hat{\epsilon}p} = \frac{E[v(\omega)] - \hat{\epsilon}v(P)p}{1-\hat{\epsilon}p}$$

This is clearly smaller than $E[v(\omega)|s = S_P] = v(P)$, thus this is the expected revenue of the auction, that is the revenue is $R^{SPA}_{peaches} = \frac{E[v(\omega)] - \hat{\epsilon}v(P)p}{1-\hat{\epsilon}p}$. We observe that the revenue continuously converges to the unconditional expectation $E[v(\omega)]$ when $\hat{\epsilon}$ converges to 0.

Next we contrasts the above with the case that the informed buyer is $\hat{\epsilon}$-informed about lemons. The revenue in the unique TRE is

$$\min_{\hat{s} \in S} E[v(\omega)|s = \hat{s}] = \min\{E[v(\omega)|s = \emptyset], E[v(\omega)|s = S_L]\}$$

Now, $E[v(\omega)|s = S_L] = v(L)$ while $E[v(\omega)|s = \emptyset] > v(L)$. Thus the revenue is $R^{SPA}_{lemons} = v(L)$, a complete collapse, *independent of how small $\hat{\epsilon}$ is!* The revenue of the seller is discontinuous in $\hat{\epsilon}$ at $\hat{\epsilon} = 0$. Note that this revenue collapse result extends to the case that the informed bidders also sometimes gets a signal about a peach, as long as he has positive probability of getting a signal about a lemon.

5.1.1 FPA vs. SPA

By comparing the SPA revenue result in Corollary 6 with the FPA revenue result in Theorem 4 of Engelbrecht-Wiggans et al. (1983), it is straightforward to show that FPA revenues are always higher than SPA revenues when only one bidder is informed.

Corollary 8. *Consider any common value domain with n agents, $n-1$ of them are uninformed, and the last agent is informed with any information structure. In any such domain the revenue of the first price auction is strictly higher than the revenue in the unique TRE of the second price auction game.*

Proof. To make the comparison, define the informed bidder's interim expected value conditional on receiving signal s as $h(s) = E[v(\omega)|s]$ and the minimum such value as $\underline{h} = \min_{s\in S} E[v(\omega)|s]$. Further, let F be the cumulative distribution function of h. According to Corollary 6, SPA revenue is at most $\underline{h}$. According to Theorem 4 of Engelbrecht-Wiggans et al. (1983), FPA revenue is

$$\int_0^\infty (1-F(h))^2 dh$$

which can be re-written as $\underline{h} + \int_{\underline{h}}^\infty (1-F(h))^2 dh$. For an informed bidder, $F(\underline{h}) < 1$ so this is clearly strictly more than $\underline{h}$. □

This result clearly implies that for both information structures we consider in Example 7 the revenue of the FPA is larger than the revenue of the unique TRE of the SPA game. For that example we can compute the revenue differences exactly. In Appendix B.1 we we use the Engelbrecht-Wiggans et al.'s (1983) revenue result for FPA and show that in both the case that the informed is $\hat{\epsilon}$-informed about lemons and the case he is $\hat{\epsilon}$-informed about peaches, the revenue of the FPA is

$$R^{FPA} = E\left[v\left(\omega\right)\right] - \hat{\epsilon}p(1-p)\left(v(P) - v(L)\right)$$

Notice that the revenue loss is proportional to $\hat{\epsilon}$, the arrival rate of cookies, regardless of whether cookies contain information about lemons or about peaches. Thus while FPA revenues are always higher than SPA revenues, the difference is substantial only when cookies identify lemons. In particular, loss in revenue from using a SPA rather than a FPA is proportional to $\hat{\epsilon}^2$ when cookies identify peaches:

$$R^{FPA}_{peaches} - R^{SPA}_{peaches} = \hat{\epsilon}^2 \frac{p^2(1-p)}{1-\hat{\epsilon}p}\left(v(P) - v(L)\right).$$

However, when cookies identify lemons, the loss is

$$R^{FPA}_{lemons} - R^{SPA}_{lemons} = \left(1 - \hat{\epsilon}(1-p)\right)p\left(v(P) - v(L)\right),$$

or approximately $p\left(v(P) - v(L)\right)$ when $\hat{\epsilon}$ is small.

We have seen that for both the case that the informed agent is $\hat{\epsilon}$-informed about lemons and the case that the informed agent is $\hat{\epsilon}$-informed about peaches, revenue of FPA does not collapse (does not tend to zero with epsilon). We next show that this is implied by a much more general observation. We observe that the revenue of FPA can be bounded from below, independent of the information structure. In Appendix B.2 we prove the following proposition.

Proposition 9. *Consider any common value domain with items of value in $[0, 1]$ and expected value of E. Assume that there are n agents, $n-1$ of them are uninformed. For any information structure for the informed agent the revenue of the FPA is at least $(E[v])^2$.*

Consider the case that items can have very low value (say 0) and that the expected value $E[v]$ is some positive constant E. This observation, in particular, says that the revenue of the FPA does not collapse to zero no matter what the information structure is, in contrast to the revenue of SPA in the unique TRE, which can be arbitrarily small if the informed agent has positive probability of getting a signal with posterior close to zero (like in the case he is $\hat{\epsilon}$-informed about lemons).

We also observed that the revenue of the FPA is continuous in $F(h)$, thus a small change in the information structure of the informed agent implies a small change in the revenue of the seller. This is in contrast to the SPA revenue, which by our result can change dramatically due to a small change in the information structure. This is exactly the case when all agents are uninformed and one of them becomes $\hat{\epsilon}$-informed about lemons. This small change in information structure have major implication on the revenue of the SPA.

5.2 Many Agents, each with Finitely Many Signals

In this section we present a condition on the information structure that allows us to generalize the result for a single informed agent (Theorem 5) to some domains in which multiple agents have informative signals. The condition is sufficient to ensure existence of a *unique* TRE. Moreover, it is a strong TRE in pure strategies. The condition on the domain, which we call "strong-high-signal", is recursively defined. In every domain that satisfies the condition there exists a signal for some agent such that his posterior belief given the signal, is independent of the signals of the others, and is as high as the posterior belief given any feasible combination of signals for all agents. Moreover, if we condition on that signal *not* being realized and consider the domain with that restriction, that domain also satisfies the condition. (A domain in which all agents are uninformed satisfies the condition.) We show that the profile of strategies in which each agent bids the posterior given his signal and the *worst* feasible combination of signals of the others, is indeed a strong TRE in pure strategies, and the unique TRE. We start by formally defining the condition.

Definition 10. *Consider a common value domain with n agents, each with finitely many signals. We say that such a domain satisfies the* strong-high-signal *property if in the domain, for some agent i, signal $s_i \in S_i$ and $s_{-i} \in S_{-i}$ such that (s_i, s_{-i}) is feasible, it holds that*

- $v(s_i, s'_{-i}) = v(s_i, s_{-i})$ *for any $s'_{-i} \in S_{-i}$ such that the profile (s_i, s'_{-i}) is feasible, and*
- $v(t) \leq v(s_i, s_{-i})$ *for any feasible profile t*

and moreover, if we consider the same domain but restricted to the case that agent i does not *receive the signal s_i, if that restricted domain contains any feasible vector of signals then it also satisfies the strong-high-signal property.*

It is easy to see that any domain with one informed agent satisfies this property, as at each point one can take the signal with the highest posterior value for the informed agent and recursively remove it. An example for a slightly more interesting domain that satisfies the strong-high-signal property is any monotonic domain with two agents, each agent i has a binary signal in $\{L_i, H_i\}$ (where H_i is the higher signal), and for which it holds that $v(H_1, H_2) = v(H_1) \geq v(H_2)$. The assumption that $v(H_1, H_2) = v(H_1)$ implies that $v(H_1, L_2) = v(H_1)$ thus the first requirement is satisfied for H_1. Now monotonicity implies that $v(H_1, H_2) \geq v(L_1, H_2) \geq v(L_1, L_2)$ thus the second requirement is satisfied. To prove that the property holds we only need to consider the domain restricted to agent 1 receiving L_1. But that domain clearly satisfies the property as it has at most one informed agent (agent 2).

We next state the main result of this section, its proof appears in Appendix C.1. We show that for a domain that satisfies the strong-high-signal property, the profile of strategies in which each agent bids the posterior given his signal and the *worst* feasible combination of signals of the others, is indeed a strong TRE in pure strategies, and the unique TRE. Observation 13 shows that a strong TRE in pure strategies might not exist if the property is violated.

Theorem 11. *Consider a common value domain with n agents, each with finitely many signals, in which the strong-high-signal property holds. Let μ be the profile of strategies in which agent i with signal $s_i \in S_i$ bids $\mu_i(s_i) = min\{v(s_i, s_{-i}) | s_{-i} \in S_{-i}$ and (s_i, s_{-i}) is $feasible\}$. Then μ is the* unique *TRE and moreover, μ is a strong TRE in pure strategies.*

This theorem has significant implications regarding the revenue of the seller in the unique TRE. In this TRE each bidder bids the posterior given his signal and the *worst* feasible combination of signals of the others, which can be much lower than the interim valuation given only the bidder's signal. For the special case of only one informed agent, the revenue equals to the lowest posterior of the informed (Corollary 6), which can be significantly lower than the expected value of the item.

The result of Theorem 11 clearly applies to the case that only one agent is informed, in this case μ is a strong TRE in pure strategies that is exactly the one described by Theorem 5. The theorem also applies to the case of monotonic domain with two agents with binary signals when $v(H_1, H_2) = v(H_1) \geq v(H_2)$. In this case the profile μ is the profile in which agent 1 getting signal H_1 bids $v(H_1)$, agent 2 getting signal H_2 bids $v(L_1, H_2)$, and each agent bids $v(L_1, L_2)$ given his low signal.

Another family of domains for which Theorem 11 applies is the family of *connected domains* which are studied by Einy et al. (2002). Connected domains are defined as follows.

Definition 12. *A domain is called a* connected domain *if the following conditions hold. Each agent i has a partition Π_i of the set of states Ω, and his signal is the element of the partition that include the realized state. The information partition Π_i of bidder i is* connected *(with respect to the common value v) if every $\pi_i \in \Pi_i$ has the property that, when $\omega_1, \omega_2 \in \pi_i$ and $v(\omega_1) \leq v(\omega_2)$ then every $\omega \in \Omega$ with $v(\omega_1) \leq v(\omega) \leq v(\omega_2)$ is necessarily in π_i. A common-value domain is* connected *(with respect to the common value) if the information partition Π_i is connected for every agent i.*

In Appendix C.2 we show that any connected domain satisfies the strong-high-signal property, thus Theorem 11 applies to any connected domain. Moreover, we observe that for connected domains the profile μ is exactly the profile of strategies pointed out by Einy et al. (2002) (the single "sophisticated equilibrium" that Pareto-dominates the rest in terms of bidders resulting utilities). We note that while connected domains allow multiple agents to have multiple signals each, there are some simple domains, even ones with a single informed bidder, that are not connected. In Appendix C.2 we also present a simple domain that satisfies the strong-high-signal property (thus Theorem 11 applies) but is not connected, and also is not equivalent to any connect domain (thus the result of Einy et al. does not apply).

The strong-high-signal property

The following observation follows form the uniqueness result presented in Theorem 18 for any domain covered by that theorem. It implies that if the strong-high-signal property if violated, the result presented in Theorem 11 might not hold.

Observation 13. *There exists a domain for which the strong-high-signal property does not hold, and for which there does not exist any strong TRE in pure strategies.*

5.2.1 Generalizing "Lemons and Peaches" to n agents

In this section we generalize the result about a single agent with either lemons or peaches information, to many agents each getting a signal for a finite (non necessarily binary) set.

Consider a monotonic common value domain with items of value in $[0, 1]$ and expected value of $E[v(\omega)]$. Assume that there are n agents, each receiving a signal s_i from an ordered, finite set of signals S_i. Let L_i and H_i denote the lowest and highest signals of agent i, respectively. Assume that the domain satisfies the conditions of Theorem 11, and consider the strong TRE μ as defined by the theorem. We next discuss the revenue of the seller in that strong TRE μ, for various information structures that generalizes the case that only one agent is informed, and is slightly informed about lemons or about peaches.

We first define what it means for an agent to be slightly *informed about peaches.* Agent i has some information about peaches if his *non-peaches* signal L_i is the common signal, has probability close to 1 (alternatively, the probability that the signal is realized to any of the other signals, the various peaches signals, is low, smaller than ϵ_i).

Definition 14. *Fix any $\epsilon_i \geq 0$. We say that agent i is ϵ_i-*informed about peaches*, if $Pr[s_i \neq L_i] = \sum_{s_i \in S_i \setminus \{L_i\}} Pr[s_i] \leq \epsilon_i$.*

We first show that if all n agents are each slightly informed about peaches, then the revenue of the SPA in the unique TRE is high (close to the social welfare, which equals to $E[v(\omega)]$).

Proposition 15. *Fix any non-negative constants $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$. Consider any monotonic domain that satisfies the strong-high-signal property and for which it also holds that every agent $i \in \{1, 2, \ldots, n\}$ is ϵ_i-informed about peaches. In the unique TRE μ (as defined in Theorem 11) the revenue of the SPA is at least*

$$E[v(\omega)] - \sum_{j=1}^{n} \epsilon_j$$

The proof of this proposition and of Proposition 17 can be found in Appendix C.3.

We next define what it means for agent i to be slightly *informed about lemons.*

Definition 16. *Fix any $\epsilon_i > 0$. We say that agent i is ϵ_i-*informed about lemons*, if*

- *$0 < Pr[s_i \neq H_i] = \sum_{s_i \in S_i \setminus \{H_i\}} Pr[s_i] < \epsilon_i$.*
- *for any $s_i \in S_i \setminus \{H_i\}$, if (s_i, s_{-i}) is feasible then $v(s_i, s_{-i}) < \epsilon_i$.*

Informally, the first assumption states that all lemon signals (not H_i, the *non-lemon* signal) for agent i are rare, the probability that any of them is realized is at most ϵ_i. The second assumption states that when i receives any one of his lemons signals it actually indicates that the value of the item, even in the best case, is very low (at most ϵ_i).

We next aim to show that when a set of agents are slightly informed about peaches and the rest of the agents are slightly informed about lemons, revenue will be very low (as long as some non-degeneracy conditions are satisfied).

Proposition 17. *Fix any positive constants $\epsilon_1, \epsilon_2, \ldots, \epsilon_i$. Consider any domain that satisfies the strong-high-signal property and for which it also holds that each agent $j \in \{1, 2, \ldots, i-1\}$ is ϵ_j-informed about peaches, while agent i is ϵ_i-informed about lemons. Assume that the domain is non degenerated in the following sense:*

- *For any $j < i$ the signal H_i does not imply L_j (alternatively, $(L_j, s_i, s_{-\{i,j\}})$ is feasible for some $s_i \neq H_i$ and some $s_{-\{i,j\}}$).*

- *For any $j > i$ and any signal $s_j \in S_j$, the signal H_i does not imply s_j (alternatively, $(s_j, s_i, s_{-\{i,j\}})$ is feasible for some $s_i \neq H_i$ and some $s_{-\{i,j\}}$).*

Then the revenue of the SPA in the unique TRE μ (as defined in Theorem 11), is at most

$$\epsilon_i + \sum_{j=1}^{i} \epsilon_j$$

An example for a domain for which Proposition 17 applies is any connected domain satisfying the following conditions. Items have uniform value in $[0,1]$. Each agent j has an increasing list of $k_j + 1$ thresholds satisfying $0 = t_j^0 < t_j^1 < t_j^3 < \ldots < t_j^{k_j} = 1$, and his signal indicates the interval between two consecutive thresholds of his that includes the realized value. Fix $i \leq n$. The condition that every agent $j < i$ is ϵ_j-informed about peaches is satisfied when $t_j^1 > 1 - \epsilon_j$. The condition that agent i is ϵ_i-informed about lemons is satisfied when $t_i^{k_i - 1} < \epsilon_i$. Every agent $j > i$ is also ϵ_i-informed about lemons when $t_i^{k_i-1} > t_j^{k_j-1}$. For such an agent j, the value conditional on his best signal is not as high as the value conditional on i's best signal (this captures the second non-degeneracy condition). It is easy to verify that the first non-degeneracy condition is satisfied for any such a domain. Proposition 17 states that the revenue is at most $\epsilon_i + \sum_{j=1}^{i} \epsilon_j$. The seller's revenue is low even though with high probability (at least $1 - \sum_{j=1}^{i} \epsilon_j$) agent i gets signal H_i and is bidding relatively high (at least $(1 - \epsilon_i - \max_{j<i} \epsilon_j)/2$). The revenue is low as all other agents are bidding low (at most ϵ_i), thus the second highest bid is low.

5.3 Two Agents, Each with a Binary Signal

When more than one agent is partially informed about the state of the world and the conditions that ensure existence of a strong TRE are not satisfied, the situation becomes much more complicated. In this section we present a complete analysis for any monotonic domain with two bidders, each getting a binary signal.

Let $\{L_1, H_1\}$ be the signals of agent 1, and $\{L_2, H_2\}$ be the signals of agent 2. Assume that the domain is monotonic and that the order of signals is $H_1 > L_1$ and $H_2 > L_2$. With some abuse of notation for any agent i with will use H_i to also denote the event that the signal of agent i was realized to H_i, and similarly for L_i. Assume without loss of generality that $v(H_1, H_2) = 1$ and that $v(L_1, L_2) = 0$. In monotonic domain it holds that $v(L_1, H_2), v(H_1, L_2) \in [0,1]$.

A domain with two bidders, each with a binary signal is *non-degenerated* if $Pr[H_1, H_2] > 0$, and for any bidder $i \in \{1,2\}$ it holds that $1 > Pr[H_i] > 0$. The main result in this section is for non-degenerated monotonic domains when none of the bidders is complete informed, yet for completeness we first discuss the rather simple cases when the domain is degenerated or at least one bidder is completely informed.

If for some bidder $i \in \{1,2\}$ it holds that $Pr[H_i] = 0$ or $Pr[H_i] = 1$ then that bidder is completely uninformed, and the results of Section 5.1 apply (unless both are completely uninformed, in that case both have a dominant strategy to bid the unconditional expectation). We are left to consider domains for which for any bidder $i \in \{1,2\}$ it holds that $1 > Pr[H_i] > 0$. A special case covered by Theorem 11 is the case when $v(H_1, H_2) = v(H_1) \geq v(H_2)$, that is, agent 1 knows the value exactly when getting H_1. In this case the unique TRE is the profile in which agent 1 getting signal H_1 bids $v(H_1)$, agent 2 getting signal H_2 bids $v(L_1, H_2)$, and each agent bids $v(L_1, L_2)$ given his low signal.

After handling all the trivial cases above, we can finally focus on the non-degenerated case when $v(H_1, H_2) > \max\{v(H_1), v(H_2)\}$. Observe that these conditions are equivalent to $1 > Pr[H_1, L_2](1 - v(H_1, L_2)) > 0$ and $1 > Pr[L_1, H_2](1 - v(L_1, H_2)) > 0$. When these two conditions hold we can assume without loss of generality that $0 < Pr[H_1, L_2](1 - v(H_1, L_2)) \leq Pr[L_1, H_2](1 - v(L_1, H_2)) < 1$ (otherwise we exchange the agents' names).

Theorem 18. *Consider any non-degenerated monotonic domain with two bidders, each with a binary signal. Assume that* $0 < Pr[H_1, L_2](1 - v(H_1, L_2)) \leq Pr[L_1, H_2](1 - v(L_1, H_2)) < 1$.

The unique TRE of the SPA game is the profile of strategies μ *in which:*

- *Every bidder* i *bids* $v(L_1, L_2) = 0$ *when getting signal* L_i.
- *Bidder* 1 *with signal* H_1 *always bids* $v(H_1, H_2) = 1$.
- *Bidder* 2 *with signal* H_2
  - *bids* $v(H_1, H_2) = 1$ *with probability* $\frac{Pr[H_1,L_2]}{Pr[L_1,H_2]} \cdot \frac{1-v(H_1,L_2)}{1-v(L_1,H_2)}$, *and*
  - *bids* $v(L_1, H_2)$ *with the remaining probability.*

Before discussing the theorem we consider the implications of this result on the revenue of the seller. As an immediate corollary we get a prediction about the revenue in the unique TRE of this game. No revenue is generated unless both bidders receive a high signal, and even in this case the revenue is only some fraction of the value created.

Corollary 19. *In any monotonic domain with* 2 *agents and binary-signal each which satisfies* $0 < Pr[H_1, L_2](1 - v(H_1, L_2)) \leq Pr[L_1, H_2](1 - v(L_1, H_2)) < 1$, *the revenue of the seller in the unique TRE of the SPA game is only*

$$Pr[H_1, H_2] \cdot \left(v(L_1, H_2) + (1 - v(L_1, H_2)) \cdot \frac{Pr[H_1, L_2]}{Pr[L_1, H_2]} \cdot \frac{1 - v(H_1, L_2)}{1 - v(L_1, H_2)} \right)$$

Note that this can be an arbitrarily small fraction of the welfare, that is the case for example when $v(H_1, L_2) = 0$ and $Pr[H_1, L_2]$ tends to 0.

We next consider the intuition behind the result. We have assumed that $Pr[H_1, L_2](1 - v(H_1, L_2)) \leq Pr[L_1, H_2](1 - v(L_1, H_2))$, how should one interpret this assumption? For bidder $i \in \{1, 2\}$ the expression $Pr[H_i, L_j](1 - v(H_i, L_j)) = Pr[H_i, L_j](v(H_1, H_2) - v(H_i, L_j))$ is the loss that bidder i has when he gets signal H_i, bids $v(H_1, H_2) = 1$ and pays his bid. The smaller this loss is, the "stronger" the bidder is. The bidder with the (weakly) smaller loss, bidder 1, is the (weakly) better informed bidder.

The theorem shows that in the unique TRE the weakly better informed agent (agent 1) is bidding 0 when getting the low signal, and bidding $v(H_1, H_2) = 1$, as if both got their high signals, when he gets his high signal. The weakly worse informed agent (agent 2) is also bidding 0 when he gets his low signal, but he does not always bid $v(H_1, H_2) = 1$ when he gets the high signal H_2. When he gets the high signal H_2 he bids $v(H_1, H_2) = 1$ with probability $\frac{Pr[H_1,L_2]}{Pr[L_1,H_2]} \cdot \frac{1-v(H_1,L_2)}{1-v(L_1,H_2)}$, and he bids $v(L_1, H_2)$ with the remaining probability. The ratio in which agent 2 is bidding $v(H_1, H_2) = 1$ is exactly the ratio between the strength of the two bidders. This ratio is 1 when the agents are ex ante symmetric, and becomes smaller and smaller as the asymmetry grows.

A particulary interesting special case is the ax ante symmetric case, $v_1 = v_2 < 1$ and $Pr[H_1, L_2] = Pr[L_1, H_2] > 0$. In this case both agents are ax ante symmetric and not completely informed about the value. In this case in the unique TRE both agents are always bidding 1 (a pure strategy) when getting the high signal, and the entire surplus when both get high signals is extracted as revenue by the seller (yet the revenue is 0 if exactly one agent gets the high signal, although the value might be positive, thus not all surplus is extracted). Thus when bidders are symmetric ex ante, our TRE refinement selects the symmetric equilibrium studied by Milgrom and Weber (1982a) and others. Hence, Milgrom and Weber's (1982a) result ranking second-price auction revenue higher than first-price auction revenue applies. (Milgrom and Weber's (1982a) result is proved for continuous signals, but the authors point out in footnote 15 that it is true more generally.) Comparing this to the result in Section 5.1 that first-price auction revenue is always higher than second-price auction revenue when only one bidder is informed illustrates that the revenue ranking depends on the level of ex ante asymmetry. While second-price auctions dominate under symmetric conditions, first-price auctions generate more revenue in sufficiently asymmetric settings.

Given ex ante asymmetry, the unique TRE identified by Theorem 18 is in mixed strategies (agent 2 is mixing between bidding 0 and bidding 1, both with positive probability) and we conclude that there is no pure TRE. Moreover, it is easy to see that the unique TRE is not a strong TRE, as one can observe that in any (ϵ, R)-tremble of the game agent 2 has negative utility by bidding 1, while bidding 0 ensures 0 utility.

We next present a sketch of the proof of Theorem 18, for the complete proof see Appendix D.2.

Sketch of the Proof of Theorem 18

We next present a sketch of the proof of Theorem 18.

Proof sketch: Fix any standard distribution R and $\epsilon > 0$ and let $\lambda(\epsilon, R)$ be the (ϵ, R)-tremble of the game. In the (ϵ, R)-tremble of the game the random bidder arrives to the auction with small probability $\epsilon > 0$ and is bidding according to a standard distribution R (its support is $[0, 1]$).

Assume that agent i with signal H_i is bidding according to distribution G_i, let g_i denote the density of G_i whenever G_i is differentiable (note that since G_i is non-decreasing it is differentiable almost everywhere, see, for example, Theorem 31.2 in Billingsley (1995)). We note that this is an abuse of notation as G_i and g_i both depend on R and ϵ.

To prove the theorem we show that for any standard distribution R and small enough ϵ a mixed NE η in each of the games $\lambda(\epsilon, R)$ exists (Lemma 69). We then show that the limit of any sequence of NE strategies in the games $\lambda(\epsilon, R)$ must converges to μ as ϵ goes to zero. Combined with the existence of a mixed NE in each of the games $\lambda(\epsilon, R)$ this show that μ is the limit of the some sequence of NE strategies in the games $\lambda(\epsilon, R)$, thus a TRE. As the limit of any sequence of NE strategies in the games $\lambda(\epsilon, R)$ must converges to μ as ϵ goes to zero, μ is the *unique* TRE.

We next present the high level arguments that prove the uniqueness of μ. Fix a standard distribution R and $\epsilon > 0$ and consider the game $\lambda(\epsilon, R)$. A NE η of the (ϵ, R)-tremble of the game λ consists of four bid distributions, one for each bidder for each signal he may receive. Thus $\eta = (G_1, G_1^L, G_2, G_2^L)$ where G_i and G_i^L are the bid distributions when $i \in \{1, 2\}$ gets the signals H_i and L_i, respectively. We first observe that if bidders never submit dominated bids, bidder $i \in \{1, 2\}$ that receives signal L_i must bid $v(L_1, L_2) = 0$. We next focus on the bids when bidder i gets the high signal H_i. To simplify the notation we denote $v_1 = v(H_1, L_2)$ and $v_2 = v(L_1, H_2)$.

For a given η we define the following notations. Let $\underline{b}_i = \inf\{b : G_i(b) > 0\}$ and $\bar{b}_i = \inf\{x : G_i(x) = 1\}$ for agent $i \in \{1, 2\}$. Define $\underline{b} = \min\{\underline{b}_1, \underline{b}_2\}$, $b_{min} = \max\{\underline{b}_1, \underline{b}_2\}$ and $b_{max} = \max\{\bar{b}_i, \bar{b}_j\}$. Note that when agent never submit dominated bids by definition it holds that $1 \geq b_{max} \geq b_{min} \geq \underline{b} \geq 0$.

We start with some necessary conditions that any mix NE η in a fixed $\lambda(\epsilon, R)$ must satisfy.

Lemma 20. *At η the following must hold.*

1. *For some $j \in \{1,2\}$ it holds that $\underline{b} = \underline{b}_j = v_j$ and $b_{min} = \underline{b}_i \geq v_i$ for $i \neq j$.*
2. *Both G_1 and G_2 are continuous and strictly increasing on (b_{min}, b_{max}). It holds that $G_1(b_{max}) = G_2(b_{max}) = 1$. Moreover, if $b_{max} > b_{min}$ then $b_{max} = \bar{b}_1 = \bar{b}_2$.*
3. *For every bidder $i \in \{1,2\}$ it holds that $G_i(b) = 0$ for every $b \in [0, \underline{b})$, and $G_i(b) = G_i(\underline{b})$ for every $b \in [\underline{b}, b_{min})$.*
4. *If $b_{min} = \underline{b}$ then $\underline{b} = \max\{v_1, v_2\}$. Additionally, if $v_1 = v_2$ then $b_{min} = \underline{b} = v_1 = v_2$ and no bidder has any atom anywhere. If $v_i > v_j$ then $b_{min} = \underline{b} = v_i$ and i has an atom at $\underline{b}$, while j has no atoms.*
5. *If $b_{min} > \underline{b}$ then for one agent, say j, it holds that $\underline{b} = \underline{b}_j = v_j$. Bidder j has an atom at v_j and bidder $i \neq j$ has an atom at*

$$b_{min} = b_i^*(G_j(v_j)) = \frac{\Pr[H_j|H_i]G_j(v_j) + v_i \Pr[L_j|H_i]}{\Pr[H_j|H_i]G_j(v_j) + \Pr[L_j|H_i]} > \max\{v_i, v_j\} \tag{2}$$

and b_{min} satisfies $b_{min} \leq v(H_i)$, and $b_{min} = v(H_i)$ if and only if $G_j(v_j) = 1$.

It also holds that either

- *$b_{max} = b_{min}$, in this case $G_i(b_{min}) = 1$, $G_j(v_j) = 1$ (j always bids v_j, i always bids b_{min}). Or*
- *$b_{max} > b_{min}$, $G_i(b_{min}) > 0$ and*

$$G_i(b_{min}) = \frac{\Pr[L_i|H_j]}{\Pr[H_i|H_j]} \frac{\int_{v_j}^{b_{min}} (x - v_j)\, \hat{r}(x)\, dx}{\hat{R}(b_{min})(1 - b_{min})} \tag{3}$$

Given the above necessary conditions we use the FOC on the bids in (b_{min}, b_{max}) to present a complete characterization of NE in the tremble of the game when ϵ is small.

Lemma 21. *If ϵ is small enough at η the following must hold. There must exist b_{min} and b_{max} such that $1 > b_{max} > b_{min} \geq 0$ and:*

- *The two bidders are symmetric ($Pr[H_1, L_2] = Pr[L_1, H_2]$ and $v_1 = v_2$) if and only if $b_{min} = \underline{b} = v_1 = v_2$ and $G_1(b_{min}) = G_2(b_{min}) = 0$ (no atoms).*
- *If $Pr[H_1, L_2](1 - v_1) = Pr[L_1, H_2](1 - v_2)$ but the bidders are not symmetric, and it holds that $v_1 > v_2$ and $Pr[H_1, L_2] < Pr[L_1, H_2]$, then bidder 1 has an atom at $b_{min} = \underline{b}_1$ of size $G_1(b_{min}) > 0$, and bidder 2 has an atom at $v_2 = \underline{b}_2 = \underline{b} < b_{min}$ of size $G_2(v_2) > 0$. It holds that*

$$b_{min} = b_1^*(G_2(v_2)) = \frac{\Pr[H_2|H_1]G_2(v_2) + v_1 \Pr[L_2|H_1]}{\Pr[H_2|H_1]G_2(v_2) + \Pr[L_2|H_1]} > \max\{v_1, v_2\} \tag{4}$$

$$G_1(b_{min}) = \frac{\Pr[L_1|H_2]}{\Pr[H_1|H_2]} \frac{\int_{v_2}^{b_{min}} (x - v_2)\, \hat{r}(x)\, dx}{\hat{R}(b_{min})(1 - b_{min})} \tag{5}$$

$$G_2(v_2) = \frac{\hat{R}(b_{max})}{\hat{R}(b_{min})} - \left(\frac{\hat{R}(b_{max})}{\hat{R}(b_{min})} - G_1(b_{min}) \right) \cdot \frac{Pr[H_1, L_2]}{Pr[L_1, H_2]} \cdot \frac{\int_{b_{min}}^{b_{max}} \frac{x - v_1}{1 - x} r(x) dx}{\int_{b_{min}}^{b_{max}} \frac{x - v_2}{1 - x} r(x) dx} \tag{6}$$

- *Assume $Pr[H_1, L_2](1 - v_1) < Pr[L_1, H_2](1 - v_2)$. Then either*
  - *$b_{min} = \underline{b}$, bidder 1 has no atom ($G_1(b_{min}) = 0$) and bidder 2 has an atom at $\underline{b} = \underline{b}_2 = v_2 \geq v_1$ of size $G_2(v_2) > 0$ specified by Equation (6), or*
  - *$b_{min} > \underline{b}$, bidder 1 has an atom at $b_{min} = \underline{b}_1$ specified by Equation (4), its size $G_1(b_{min}) > 0$ is specified by Equation (5), and bidder 2 has an atom at $v_2 = \underline{b}_2 = \underline{b} < b_{min}$ of size $G_2(v_2) > 0$ specified by Equation (6).*

Moreover, it always hold that

$$G_1(b) = \begin{cases} 0 & \text{if } 0 \leq b < b_{min}; \\ \frac{Pr[L_1|H_2]}{Pr[H_1|H_2]} \cdot \frac{\epsilon}{\hat{R}(b)} \cdot \int_{b_{min}}^{b} \frac{x - v_2}{1 - x} r(x)dx + G_1(b_{min}) \cdot \frac{\hat{R}(b_{min})}{\hat{R}(b)} & \text{if } b_{min} \leq b \leq b_{max}; \\ 1 & \text{if } b_{max} < b \leq 1. \end{cases}$$

and

$$G_2(b) = \begin{cases} 0 & \text{if } 0 \leq b < v_2; \\ G_2(v_2) & \text{if } v_2 \leq b < b_{min}; \\ \frac{Pr[L_2|H_1]}{Pr[H_2|H_1]} \cdot \frac{\epsilon}{\hat{R}(b)} \int_{b_{min}}^{b} \frac{x - v_1}{1 - x} \cdot r(x)dx + G_2(v_2) \cdot \frac{\hat{R}(b_{min})}{\hat{R}(b)} & \text{if } b_{min} \leq b \leq b_{max}; \\ 1 & \text{if } b_{max} < b \leq 1. \end{cases}$$

From this we derive that $G_2(v_2)$ approaches $1 - \frac{Pr[H_1, L_2]}{Pr[L_1, H_2]} \cdot \frac{1 - v(H_1, L_2)}{1 - v(L_1, H_2)}$ when ϵ goes to 0, and that $G_1(b_{min})$ tends to 0. Moreover, we derive that for any large enough bid $b \in (0, 1)$ both $G_1(b)$ and $G_2(b) - G_2(v_2)$ are bounded from above, by some function that tends to 0 as ϵ goes to 0. We conclude that the limit of the sequence of these mixed NE is exactly μ, as claimed in the theorem statement.

6 Discussion: Mechanism Design

The previous section shows that in the common value model the revenue of the SPA might be significantly smaller than the welfare. In the section we consider the problem of maximizing the revenue the seller receives by selling the item.

In the common value model there is a trivial mechanism that is ex-ante individually rational and maximizes the welfare as well as the revenue: we offer the first buyer a take-it-or-leave-it offer to buy the item for the price equal to the unconditional expectation of the item.

Yet this trivial mechanism does not extend to the case that there is some private component to the value of the item. For example, in the domain of online advertisement it is reasonable to assume that an informed buyer (advertiser) that has a high quality signal (cookie) on the user machine can tailor a specific advertisement to the specific user, generating some additional value over the common value created by placing a generic advertisement that is not user specific.

This motivates us to consider the following generalization of the model with a single informed bidder, in which the informed bidder is also *advantaged*. In this model there are n potential buyers. One random buyer i is informed about the state of the world (gets a signal $s_i \in S_i$), while the others are uninformed. Assume that the signals are sorted by the expected common value to an uninformed bidder. Assume that for the maximal signal s_{max} the value for the informed bidder is larger than the common value by some $B > 0$ (this is his Bonus). Let s_{min} denote the lowest signal.

Let E be the unconditional expected value of the item to an uninformed bidder. Let p_{max} be the probability of the signal to be realized to s_{max}, and let L be the expected value of the item conditional on the signal being s_{min}. The expected social welfare when the realized informed bidder always gets the item is $E + p_{max}B$. In this model selling the item ax-ante to a fixed agent at his expected value will generate revenue of $E + \frac{p_{max}B}{n}$, which can be significantly lower than the maximal social welfare.

A mechanism that gets revenue that equals the maximal welfare must allocate the item efficiently. Running the second price auction in this scenario will indeed maximize the social welfare. Yet, one can easily extend Theorem 5 to this model and see that for any realized informed bidder the unique TRE in this model is exactly the same as the one described by the theorem (with the adjustment that the informed bidder with signal s_{max} bids his value that includes the bonus).

Yet, we can build a mechanism that is ex-ante individually rational, is socially efficient and extracts (almost) the entire welfare as revenue. All this in the *unique* outcome of the mechanism under our refinement as we explain below.

The mechanism has two stages. The mechanism first presents each bidder with a take-it-or-leave-it offer to buy the right to bid in a second price auction (SPA), and then runs a SPA with the bids of every agent that has bought the right to enter the SPA. Theorem 5 (and its extension to this model) predicts a *unique* TRE. The payment in the SPA is always going to be L. The take-it-or-leave-it price is set to be slightly less than the expected utility that the agent gets by participating in the SPA, assuming all agents participate in the SPA and bid according to the *unique* TRE in that game. The entry price is set to be slightly less than $(E + p_{max}B - L)/n$.

As TRE provides a *unique* prediction to the outcome of the second stage, agents have a *unique* rational decision when facing the entry decision, and they choose to pay the entry fee. Thus, in the *unique* subgame-perfect-equilibrium that uses the TRE refinement, agents will all choose to enter (pay the entry fee), and allocation will be socially efficient in the SPA. The utility of each agent is essentially 0 (his gain goes to 0 as the entry price tends to $(E + p_{max}B - L)/n$). Although the revenue in the SPA is low, the entire utility an agent gets in this auction in expectation is essentially charged as entry fee. The revenue from entry would be $n(E + p_{max}B - L)/n = E + p_{max}B - L$, while the revenue in the SPA would be L, and the total revenue is exactly the social welfare $E + p_{max}B$. Thus the total revenue essentially equals to the social welfare.

The above mechanism can only be used when agent can reasonably predict the outcome of the SPA that takes place at the second stage, and it can be extended to any other scenario in which a uniqueness result can be proven about the outcome of the SPA game under some solution concept.

Interim Individually Rational Mechanism

We note that the above mechanism is not interim individually rational. We next consider the problem of designing an interim individually rational mechanism for this setting, when the informed player has only two signals s_{min} and s_{max}. The mechanism we design is dominant strategy incentive compatible. Let L be the value conditioned on s_{min} and $P + B$ be the value of the advantaged bidder conditioned on s_{max}.

While our model is not one of independent private value, it is sufficiently close that it seems useful to consider the optimal auction when each players value is sampled independently and identically from the following distribution: the value is L with probability $1 - 1/n$, and $P + B$

with probability $1/n$. For this instance, Myerson's optimal auction is to have some reserve price r and some floor price f. If some bidders bid at least r then we run a second price auction with reserve r, otherwise we randomly choose a winner among those who bid at least f and charge the winner f.

In our model we can indeed set $f = L$ and $r = P + B - z$, where $z = (P + B - f)/n$ is the expected utility of agent i bidding f given signal s_{max} (conditioned on every other agent j bidding f).

The revenue obtained is $(1 - p_{max})f + p_{max}r$. Note that this is at least $(1 - 1/n)-$fraction of the social welfare which is $(1 - p_{max})L + p_{max}(P + B)$.

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A Multiplicity of Equilibria under Perfect Equilibrium

Considering refinements for our game, one natural candidate is Selten's (1975) Tremble Hand Perfect Equilibrium (PE). In this section we show that in our common value SPA with asymmetric information, PE does *not* provide the natural unique prediction one would expect in the most basic setting with two agents: one informed agent with a binary signal, and one uninformed agent. Note that in this setting there is a unique TRE in undominated strategies, and it is a strong TRE in pure strategies. In this natural equilibrium, the informed bids his posterior value while the uninformed bids to match the lowest possible bid of the informed.

Formally, consider the setting with two agents, one informed agent with a binary signal, and one uninformed agent. Assume that the common value is 0 conditional on the informed low signal, and 1 conditional on his high signal. Each signal is realized with probability 1/2. Each agent's action space (bid space) is the set $[0, 1]$ (an infinite set). In the unique TRE in undominated strategies the informed bids 0 on the low signal and 1 on the high signal, while the uninformed always bids 0.

We note that PE is usually defined for *finite normal form games* while our game is a game of incomplete information with infinite strategy spaces (finite type spaces but infinite action spaces). The adaptation of the solution concept to incomplete information is relatively straightforward. The move to infinite games is more delicate and we discuss two adaptations that were suggested in Simon and Stinchcombe (1995) (extending these adaptations to the incomplete information setting) and show that neither provide a unique prediction.

We start by presenting Simon and Stinchcombe's (1995) reformulation Selten's (1975) Tremble Hand Perfect Equilibrium (PE) for finite (normal form) games with complete information. Let N be a finite set of agents. For agent $i \in N$ let A_i be a finite set of pure actions, and let $A = \times_{i \in N} A_i$. Let Δ_i (resp. Δ_i^{fs}) be the set of probability distributions (resp. full support probability distributions) on A_i. Let $\Delta = \times_{i \in N} \Delta_i$ and $\Delta^{fs} = \times_{i \in N} \Delta_i^{fs}$. For $\mu \in \Delta$, let $Br_i(\mu_{-i})$ denote i's set of mixed-strategy best-responses to the vector of strategies of the others μ_{-i}.

Definition 22. *(Selten (1975)) Consider a finite game. Fix $\epsilon > 0$. A vector $\mu^\epsilon = (\mu_i^\epsilon)_{i \in N}$ in Δ^{fs} is an* ϵ-Perfect Equilibrium *if for each agent $i \in N$ it holds that*[8]

$$d_i(\mu_i^\epsilon, Br_i(\mu_{-i}^\epsilon)) < \epsilon$$

where $d_i(\mu_i, \nu_i) = \sum_{a_i \in A_i} |\mu_i(a_i) - \nu_i(a_i)|$.

A vector $\mu = (\mu_i)_{i \in N}$ in Δ is a Perfect Equilibrium *if there exists an infinite sequence of positive numbers $\epsilon_1, \epsilon_2, \ldots$ which converges to 0 such that (1) for each j, μ^{ϵ_j} is an ϵ_j-Perfect Equilibrium and (2) for every $i \in N$ it holds that $\mu_i^{\epsilon_j}$ converges in distribution to μ_i when j goes to infinity.*

Loosely speaking, for a finite (normal form) game a Perfect Equilibrium is a limit, as ϵ goes to 0, of a sequence of full support strategy vectors, each element of such a vector is ϵ close to being a best response to the other agent's strategies in that element of the sequence of strategy vectors.

We next discuss two adaptations, suggested in (Simon and Stinchcombe 1995), of PE to infinite games. The first is called *"limit-of-finite"* which considers the limit of a sequence of

[8]Informally, his strategy is at most ϵ away from being a best response.

strategies in a sequence of finite games, in each game only a finite subset of actions is allowed and every player's strategy has full support. The distance from every action to the set of allowed actions goes to zero and the sequence of strategies converges to the "limit-of-finite". The second is called *strong perfect equilibrium* which looks directly at the infinite game and requires strictly positive mass to every nonempty open subset and the sequence of strategies converges to the strong perfect equilibrium.

Next, we adjust these concepts to games with incomplete information, finite types spaces but infinite action spaces, and show that neither predict a unique equilibrium in the simple setting discussed above.[9]

A.1 Limit of Finite Games

We next define the notion of *limit-of-finite* Perfect Equilibrium for games with incomplete information, finite types spaces but infinite action spaces. The approach is to define perfect equilibrium as the limit of ϵ-perfect equilibria for sequences of successively larger (more refined) finite games.

Let N be a finite set of agents. For agent $i \in N$ let T_i be a finite set of types for agent i. Assume that the agents have a common prior over types. Let A_i be a compact (infinite) set of actions. Let B_i be a nonempty finite subset of A_i, and let $B = \times_{i \in N} B_i$. For such a B_i, let $\Delta_i(B_i)$ (resp. $\Delta_i^{fs}(B_i)$) be the set of probability distributions (resp. full support probability distributions) on B_i.

A B_i-supported mixed strategy $\mu_i(B_i)$ for agent i is a mapping from his type t_i to an element of $\Delta_i(B_i)$. For a profile of mixed strategies $\mu(B) = (\mu_i(B_i))_{i \in N}$, agent i and type $t_i \in T_i$, let $Br_i^{t_i}(B_i, \mu_{-i})$ denote i's set of B_i-supported mixed-strategy best-responses to the vector of strategies of the others $\mu_{-i}(B_{-i})$ (with respect to the given prior and the utility functions) when his type is t_i.

Definition 23. *Consider a game with incomplete information, finite types spaces but infinite action spaces. Fix $\epsilon > 0$ and $\delta > 0$. For each agent $i \in N$ let B_i^δ denote a finite subset of A_i within (distance) δ of A_i. A vector $\mu^{(\epsilon,\delta)} = (\mu_i^{(\epsilon,\delta)})_{i \in N}$ such that for each i and $t_i \in T_i$ it holds that $\mu_i^{(\epsilon,\delta)}(t_i) \in \Delta_i^{fs}(B_i^\delta)$ is an (ϵ,δ)-*Perfect Equilibrium *if for each agent $i \in N$ and type $t_i \in T_i$ it holds that*

$$d_i^\delta(\mu_i^{(\epsilon,\delta)}(t_i), Br_i^{t_i}(B_i^\delta, \mu_{-i}^{(\epsilon,\delta)})) < \epsilon$$

where $d_i^\delta(\mu_i, \nu_i) = \sum_{a_i \in B_i^\delta} |\mu_i(a_i) - \nu_i(a_i)|$.

A vector $\mu = (\mu_i)_{i \in N}$ is a limit-of-finite Perfect Equilibrium *if there exists two infinite sequences of positive numbers $\epsilon_1, \epsilon_2, \ldots$ and $\delta_1, \delta_2, \ldots$ both converging to 0 such that (1) for each j, $\mu^{(\epsilon_j,\delta_j)}$ is an (ϵ_j, δ_j)-Perfect Equilibrium and (2) for every $i \in N$ and $t_i \in T_i$ it holds that $\mu_i^{(\epsilon_j,\delta_j)}(t_i)$ converges in distribution to $\mu_i(t_i)$ when j goes to infinity.*

We next show that there are multiple strong PE in the infinite game with one informed agent with a binary signal and one uninformed agent.

[9] We note that with tremble that is *independent* of the signal of the informed agent, such multiplicity of equilibria result cannot be proven. Yet, the unique equlibirum that is the result of any such tremble is *not* the one we would expect. In the same setting of an item of a common value 0 or 1, with equal probability, and two agents, one perfectly informed and one uninformed, we observe the following. For any tremble of the informed that is independent of the informed agent's signal, the best response of the uninformed agent is to bid the unconditional expectation (half) as this is the value of the item conditional on winning in the case the informed trembles (and if he does not, the uninformed agent just pays the exact value of the item if winning, as the price is set by the informed agent).

Proposition 24. *Consider the infinite game with one informed agent with a binary signal and one uninformed agent as defined above. For any $y \in (0, 1)$, the following is a (pure strategy) limit-of-finite perfect equilibrium in this infinite game: The informed bids according to his dominant strategy (his posterior: 0 on low signal, 1 on high signal), while the uninformed always bids y.*

Proof. Consider the following natural way to make our game finite by discretizing the bids: fix a large natural number m and only allow bids of the form k/m for $k \in \{0, 1, \ldots, m\}$. Note that as m grows to infinity the distance between any bid y and such a set of bids decreases to zero.

Fix $\epsilon > 0$ that is small enough. Fix m that is large enough and fix $k_0 \in \{1, \ldots, m-1\}$ such that $(k_0 + 1)/m$ has minimal distance to y out of all bids of form k/m. To prove the claim we present a profile of strategies with full support over the discrete set of bids that is close to the profile in which the informed bids according to his dominant strategy while the uninformed always bids y. The strategies that we build have an atom of size at least $1 - \epsilon$ on the specified bids. For the informed with low signal, the probability on every bid other than 0 is proportional to ϵ^2, while for the informed with high signal the probability of every bid other than 1 is proportional to ϵ^3, except for k_0/m for which he assigns probability of about ϵ. This motivates the uninformed to bid $(k_0 + 1)/m$, right above this "gift" given by the informed bidder with high signal, and we show that such a bid is his best response. We next define the strategies formally.

The informed agent with low signal is bidding 0 with probability $1 - \epsilon^2$, and for any $k \in \{1, \ldots, m\}$ he bids k/m with probability ϵ^2/m. The informed agent with high signal is bidding 1 with probability $1 - \epsilon$. He bids k_0/m with probability $\epsilon - \epsilon^3$, and for any $k \in \{0, \ldots, m-1\}$ such that $k \neq k_0$, he bids k/m with probability $\epsilon^3/(m-1)$.

The uninformed agent is bidding $(k_0 + 1)/m$ with probability $1 - \epsilon$, and for any $k \in \{0, \ldots, m\}$ such that $k \neq k_0 + 1$ he bids k/m with probability ϵ/m.

The informed agent has a dominant strategy to bid his posterior value, and his strategy is clearly ϵ close to that strategy. It remains to show that the strategy of the uninformed is ϵ close to his best response (to the strategy of the informed). We claim that if ϵ is small enough the best response of the uninformed to the strategy of the informed is to bid $(k_0 + 1)/m$ with probability 1. Indeed, consider any bid j/m:

- If $j = k_0 + 1$ then the informed has positive utility as when the value is high he has utility of at least $1/m$ with probability at least $(\epsilon - \epsilon^3)$. When the value is low his loss is at most $(k_0 + 1)/m$ and this happens only with probability at most ϵ^2. For small enough ϵ the loss will be smaller than the gain.
- If $j = 0$ then the uninformed has utility 0.
- If $0 < j < k_0$ then the uninformed wins item of value 1 with probability at most $j\epsilon^3/(2 \cdot (m-1))$ (as the quality is high with probability 1/2 and in such case he only wins if the informed is bidding below him), thus his expected value is at most $j\epsilon^3/(2 \cdot (m-1))$. On the other hand his expected payment is at least $(1/4) \cdot (\epsilon^2/m) \cdot (1/m)$ (in case it is low value he pays at least $1/m$ with probability $(1/2) \cdot (\epsilon^2/m)$ - the probability of the other bidding $1/m$ and tie is broken in favor of him). Thus his expected utility is at most $j\epsilon^3/(2 \cdot (m-1)) - \epsilon^2/4m^2$ which is negative for small enough $\epsilon > 0$.
- If $j = k_0$ then we claim that this bid is dominated by bidding $(k_0 + 1)/m$. Due to random tie breaking the bid of k_0/m only wins half of the times when the value is high and the informed is also bidding k_0/m. By increasing his bid to $(k_0 + 1)/m$ the uninformed will

always win in this case. The affect of this change is linear in ϵ. The negative effect due to winning more when the informed gets the low signal is only of the order of ϵ^2, thus for small enough ϵ it will be smaller.

- If $j > k_0 + 1$ then we claim that this bid is dominated by bidding $(j-1)/m$. This follow since the probability of winning high value items decreases by order of ϵ^3, while the probability of not paying for low value items decreases by order of ϵ^2.

□

Note that the proof of the proposition shows that PE does not provide a unique prediction even if we consider finite discrete action spaces. This seems to indicate that the problem with PE (with respect to our setting) is deeper than just its extension to games with infinite action spaces.

A.2 Strong Perfect Equilibrium

We next define the notion of *strong* Perfect Equilibrium for games with incomplete information, finite types spaces but infinite action spaces. Let N be a finite set of agents. For agent $i \in N$ let T_i be a finite set of types for agent i. Assume that the agents have a common prior over types. Let A_i be a compact (infinite) set of actions. Let Δ_i be the set of probability measures on A_i, while Δ_i^{fs} be the set of probability measures on A_i assigning strictly positive mass to every nonempty open subset of A_i. We measure the distance between two measures μ, ν on an infinite actions space using the following metric:

$$\rho(\mu, \nu) = \sup\{|\mu(B) - \nu(B)| \ : B \text{ } measurable\}$$

A mixed strategy μ_i for agent i is a mapping from his type $t_i \in T_i$ to an element of Δ_i. For a profile of mixed strategies $\mu = (\mu_i)_{i \in N}$ agent i and type $t_i \in T_i$, let $Br_i^{t_i}(\mu_{-i})$ denote i's set of mixed-strategy best-responses to the vector of strategies of the others μ_{-i} (with respect to the given prior and the utility functions) when his type is t_i.

Definition 25. *Consider a game with incomplete information, finite types spaces but infinite action spaces. Fix $\epsilon > 0$. A vector $\mu^\epsilon = (\mu_i^\epsilon)_{i \in N}$ such that for each i and $t_i \in T_i$ it holds that $\mu_i(t_i) \in \Delta_i^{fs}$ is a* strong ϵ-Perfect Equilibrium *if for each agent $i \in N$ and type $t_i \in T_i$ it holds that*

$$\rho_i(\mu_i^\epsilon(t_i), Br_i^{t_i}(\mu_{-i}^\epsilon)) < \epsilon$$

A vector $\mu = (\mu_i)_{i \in N}$ is a strong Perfect Equilibrium *if there exists an infinite sequence of positive numbers $\epsilon_1, \epsilon_2, \dots$ which converges to 0 such that (1) for each j, μ^{ϵ_j} is a strong ϵ_j-Perfect Equilibrium and (2) for every $i \in N$ and $t_i \in T_i$ it holds that $\mu_i^{\epsilon_j}(t_i)$ converges in distribution to $\mu_i(t_i)$ when j goes to infinity.*

We next show that there are multiple strong PE in the infinite game with one informed agent with a binary signal and one uninformed agent. The construction of the strategies in the next proposition is very similar to the one in Proposition 24.

Proposition 26. *Consider the infinite game with one informed agent with a binary signal and one uninformed agent as defined above. For any $y \in (0, 1)$, the following is a (pure strategy) strong perfect equilibrium in this infinite game: The informed bids according to his dominant strategy (his posterior: 0 on low signal, 1 on high signal), while the uninformed always bids y.*

Proof. Fix some $y \in (0,1)$. Consider the following tremble for a given $\epsilon > 0$ that is small enough.

The informed agent with low signal is bidding with CDF $F_L(x) = 1-\epsilon^2+x\epsilon^2$ for $x \in [0,1]$. (He bids 0 with probability $1-\epsilon^2$ or uniformly between 0 and 1 with probability ϵ^2.)

The informed agent with high signal is bidding with CDF F_H: For $x \in [0, y-\epsilon]$ it holds that $F_H(x) = x\epsilon^3$. For $x \in (y-\epsilon, y]$ it holds that $F_H(x) = F_H(y-\epsilon)+(x-y+\epsilon)(1-\epsilon^2)$. For $x \in (y,1)$ it holds that $F_H(x) = F_H(y)+(x-y)\epsilon^3$, and finally, $F_H(1) = 1$. (He bids 1 with probability $1-\epsilon+\epsilon^4$, uniformly between $y-\epsilon$ and y with probability $\epsilon-\epsilon^3$, and uniformly over all other bids in $[0,1]$ with the remaining probability $\epsilon^3(1-\epsilon)$.)

The uninformed agent is bidding with CDF G: For $x \in [0,y)$ it holds that $G(x) = x\epsilon$. For $x = y$ it holds that $G(x) = G(y) = y\epsilon+1-\epsilon$. For $x \in (y,1]$ it holds that $G(x) = G(y)+(x-y)\epsilon$. (He bids y with probability $1-\epsilon$ or uniformly between 0 and 1 with probability ϵ.)

Clearly these strategies have full support and their limit as ϵ goes to 0 is as required.

The informed agent has a dominant strategy to bid his posterior value, and his strategy is clearly ϵ close to that strategy. It remains to show that the strategy of the uninformed is ϵ close to his best response (to the strategy of the informed). We claim that if ϵ is small enough the best response of the uninformed to the strategy of the informed is to bid y with probability 1. Indeed, consider any bid z:

- If $z = 0$ then the agent has utility 0.
- If $z = y$ then for small enough $\epsilon > 0$ the agent has positive utility. Indeed his expected gain from high value items is at least $1/2 \cdot F_H(y)(1-y) = (\epsilon-\epsilon^3(1-y+\epsilon))(1-y)/2 \geq c\epsilon$ for some constant $c > 0$ (for small enough $\epsilon > 0$), while his expected loss from low value items is at most $1/2 \cdot (1-F_L(0))y \leq (y/2)\epsilon^2 \leq \epsilon^2$.
- If $0 < z < y$ then for small enough $\epsilon > 0$ it holds that $0 < z < y-\epsilon$. Moreover, for small enough $\epsilon > 0$ the agent has negative utility. Indeed his expected gain is at most $1/2 \cdot F_H(z) \cdot 1 \leq z\epsilon^3$, while his expected loss is at least $1/2 \cdot (F_L(z)-F_L(z/2)) \cdot z/2 \geq z^2\epsilon^2/4$.
- If $z > y$ then for small enough $\epsilon > 0$ the agent can increase his utility by bidding y instead of bidding z. Indeed his expected loss of value by bidding y instead of z is at most $1/2 \cdot (F_H(z)-F_H(y)) \cdot 1 = (y-z)\epsilon^3/2$, while his expected reduction in payment is at least $1/2 \cdot (F_L(z)-F_L(y)) \cdot y \geq (z-y)\epsilon^2/2$.

□

B One Informed Agent

B.1 FPA Revenue in Example 7

Let $h = E\left[v\left(w\right)|s\right]$ be the informed bidder's interim value given signal s, and F be its cumulative distribution. Then by Engelbrecht-Wiggans et al.'s (1983) Theorem 4, FPA revenue is $\int_0^\infty \left(1-F\left(h\right)\right)^2 dh$. First consider the peaches case. Let $E^- = \frac{E[v(w)]-\hat{\epsilon}pv(P)}{1-\hat{\epsilon}p}$ be the posterior given the signal $\emptyset$. As shown in the main text, $h \in \{E^-, v\left(P\right)\}$ and $\Pr\left(h = v\left(P\right)\right) = \hat{\epsilon}p$. Therefore

$$F_{peaches}\left(h\right) = \begin{cases} 0, & h < E^- \\ 1-\hat{\epsilon}p, & h \in [E^-, v\left(P\right)) \\ 1, & h \geq v\left(P\right) \end{cases},$$

and hence

$$R^{FPA}_{peaches} = \left(1 - (\hat{\epsilon}p)^2\right) \frac{E\left[v\left(w\right)\right] - \hat{\epsilon}pv\left(P\right)}{1 - \hat{\epsilon}p} + (\hat{\epsilon}p)^2 \, v\left(P\right) = E\left[v\left(w\right)\right] - \hat{\epsilon}p\left(1-p\right)\left(v\left(P\right) - v\left(L\right)\right).$$

Second, consider the lemons case. Let $E^{+} = \frac{E[v(w)] - \hat{\epsilon}(1-p)v(L)}{1-\hat{\epsilon}(1-p)}$ be the posterior given the signal $\emptyset$. Now, $h \in \{v\left(L\right), E^{+}\}$ and $\Pr\left(h = v\left(L\right)\right) = \hat{\epsilon}\left(1-p\right)$. Therefore

$$F_{lemons}\left(h\right) = \begin{cases} 0, & h < v\left(L\right) \\ \hat{\epsilon}(1-p), & h \in \left[v\left(L\right), E^{+}\right) \\ 1, & h \geq E^{+} \end{cases},$$

and hence

$$\begin{aligned} R^{FPA}_{lemons} &= v(L) + \left(1 - \hat{\epsilon}\left(1-p\right)\right)^2 \left(E^{+} - v(L)\right) \\ &= E\left[v\left(w\right)\right] - \hat{\epsilon}p\left(1-p\right)\left(v\left(P\right) - v\left(L\right)\right). \end{aligned}$$

B.2 Bounding the FPA Revenue from Below

In this section we prove Proposition 9.

Proposition 27. *Consider any common value domain with items of value in* $[0,1]$ *and expected value of* E*. Assume that there are* n *agents,* $n-1$ *of them are uninformed. For any information structure for the informed agent the revenue of the FPA is at least* E^2*.*

Proof. Define the informed bidder's interim expected value conditional on receiving signal s as $h(s) = E[v(\omega)|s]$. Further, let F be the cumulative distribution function of h. Note that as items have value in $[0,1]$, $h \in [0,1]$ and $F(1) = 1$. According to Theorem 4 of Engelbrecht-Wiggans et al. (1983), FPA revenue is

$$\int_0^1 (1 - F(h))^2 dh$$

and the informed agent expected profit is

$$\int_0^1 F(h)(1 - F(h))dh$$

Note that the revenue and the informed agent's profit sum up to E, the expected value of the item (the social welfare). To bound the revenue from below we bound the informed agent's profit from above. We use the following result due to Ahlswede and Daykin (1979).

Lemma 28. *If, for* 4 *non-negative functions* g_1, g_2, g_3, g_4 *mapping* $\mathbb{R} \to \mathbb{R}$*, the following holds:*

$$\text{for all } x, y \in \mathbb{R}, \; g_1(\max(x,y)) \cdot g_2(\min(x,y)) \geq g_3(x) \cdot g_4(y),$$

then it follows that

$$\int_a^b g_1(t)dt \cdot \int_a^b g_2(t)dt \geq \int_a^b g_3(t)dt \int_a^b g_4(t)dt.$$

We apply this lemma by setting

$$g_1(t) = F(t),\ g_2(t) = 1 - F(t),\ g_3(t) = F(t) \cdot (1 - F(t)),\ g_4(t) = 1.$$

Monotonicity of F implies that the conditions of the lemma hold. Indeed, if $x'' > x'$,

$$F(x'') \cdot (1 - F(x')) \geq F(x'') \cdot (1 - F(x''))$$

and

$$F(x'') \cdot (1 - F(x')) \geq F(x') \cdot (1 - F(x')),$$

.

Then, it follows that

$$E \cdot (1 - E) = \int_0^1 F(t)dt \cdot \int_0^1 (1 - F(t))dt \geq \int_0^1 F(t)(1 - F(t))dt.$$

As the revenue equals to the welfare minus the informed agent's profit we conclude that the revenue is bounded from above by E^2:

$$\int_0^1 (1 - F(h))^2 dh = E - \int_0^1 F(t)(1 - F(t))dt \leq E^2$$

□

C Many Agents, each with Finitely Many Signals

C.1 Proof of Theorem 11

We prove Theorem 11 by induction, and we begin with an observation and a lemma that are applied at each induction step. First, recall that Theorem 11 defines:

$$\mu_i(s_i) \equiv min\{v(s_i, s_{-i})|s_{-i} \in S_{-i}\ and\ (s_i, s_{-i})\ is\ feasible\}.$$

A natural binary relation between signals can be defined using the relation between the lower bounds they place on the expected value. We say that signal s_i of bidder i is *weakly lower* than signal s_j of bidder j if $\mu_i(s_i) \leq \mu_j(s_j)$, and is *strictly higher* than signal s_j of bidder j if $\mu_i(s_i) > \mu_j(s_j)$.

The next lemma is a major step in showing that bidder i with signal s_i does not bid *above* $\mu_i(s_i)$.

Lemma 29. *Fix a signal s_j received by bidder j and any strategy profile η in which every bidder i with signal s_i strictly higher than s_j ($\mu_i(s_i) > \mu_j(s_j)$) bids $\mu_i(s_i)$ with probability* 1.

1. *If η is a NE of the tremble $\lambda(\epsilon, R)$ with $\epsilon > 0$, then no bidder i with signal s_i (including bidder j with signal s_j) weakly lower than s_j bids strictly above $\mu_j(s_j)$.*
2. *In the original game λ and in any tremble $\lambda(\epsilon, R)$ for $\epsilon > 0$, the utility of bidder j with signal s_j from bidding $\mu_j(s_j)$ is at least as high as his utility from any higher bid.*

Proof. **Proof of part (1)**: Let $\bar{b}_i(s_i)$ be the supremum bid by bidder i with signal s_i. Let $\bar{b}$ be the maximum supremum bid among signals weakly lower than s_j:

$$\bar{b} \equiv \max_{i \in N, s_i \in S_i} \{\bar{b}_i(s_i) : \mu_i(s_i) \leq \mu_j(s_j)\}.$$

Suppose η is a NE of the tremble $\lambda(\epsilon, R)$ but $\bar{b} > \mu_j(s_j)$. Let $\delta > 0$ be sufficiently small such that (1) $\mu_j(s_j) < \bar{b} - \delta$ and (2) for any bidder i and signal s_i if $\bar{b}_i(s_i) < \bar{b}$ implies $\bar{b}_i(s_i) < \bar{b} - \delta$. With positive probability, no signal strictly higher than s_j is realized and the high bid falls in the interval $(\bar{b} - \delta, \bar{b}]$. Therefore at least one bidder k with a signal s_k that satisfies $\bar{b}_k(s_k) = \bar{b}$ and $\mu_k(s_k) \leq \mu_j(s_j)$ (possibly $k = j$ and $s_k = s_j$) wins with positive probability with a bid in the interval $(\bar{b} - \delta, \bar{b}]$.

Fix any bid $b \in (\bar{b} - \delta, \bar{b}]$ that wins with positive probability conditional on being placed by bidder k with signal s_k. We show below that for bidder k with signal s_k, bidding $\mu_j(s_j)$ is strictly more profitable than bidding b. Because bidder k with signal s_k makes such bids with positive probability, this contradicts η being a NE. The argument follows below.

Consider a particular realization in which bidder k receives signal s_k. Let b_{-k}^{max} be the highest realized bid of bidders other than k (including the random bidder). Further, let bidder i be the bidder who has the highest realized signal s_i and his bid be b_i. (If there are multiple bidders whose signals tie for the highest then choose any i from the set.)

Now compare k's outcome from bidding b rather than $\mu_j(s_j)$. If $b_{-k}^{max} < \mu_j(s_j)$ or $b_{-k}^{max} > b$ then k's outcome and payoff are unchanged by bidding b rather than $\mu_j(s_j)$. However, if $b_{-k}^{max} \in [\mu_j(s_j), b]$ then k wins and pays b_{-k}^{max} by bidding b at some cases were he was losing by bidding $\mu_j(s_j)$. Consider three cases. **First**, suppose that some bidder with a signal strictly higher than s_j is bidding b_{-k}^{max}. Then by assumption $b_{-k}^{max} = b_i = \mu_i(s_i)$ and by the strong-high-signal property (SHSP) $v(s) = \mu_i(s_i)$. Thus the additional win is priced at its value and does not change k's payoff. **Second**, suppose $b_{-k}^{max} = \mu_j(s_j)$. Then by SHSP $v(s) \leq \mu_j(s_j)$ and the additional win is priced at or above its value and weakly reduces k's payoff. **Third**, suppose $b_{-k}^{max} \in (\mu_j(s_j), b]$ and it is not the bid of a bidder with a strictly higher signal. If no signal strictly higher than s_j is realized, then by SHSP $v(s) \leq \mu_j(s_j)$. If at least one signal strictly higher than s_j is realized, then by assumption $b_{-k}^{max} > b_i = \mu_i(s_i)$ and by SHSP $v(s) = \mu_i(s_i)$. In either case, the additional win must be priced strictly above its value ($b_{-k}^{max} > v(s)$) and strictly reduces k's payoff.

The preceding paragraph shows that for any realization, bidding b yields a weakly lower payoff for k than bidding $\mu_j(s_j)$ and in the third case yields a strictly lower payoff. The third case occurs with positive probability in any tremble $\lambda(\epsilon, R)$ with $\epsilon > 0$. Therefore bidding b rather than $\mu_j(s_j)$ strictly reduces k's expected payoff ex ante.

Proof of part (2): In the proof of part (1) above, we showed that (for any realization) bidding b yields a weakly lower payoff for bidder k with signal s_k than bidding $\mu_j(s_j)$. The same argument can be repeated under the assumptions of part (2) to show that bidding $\mu_j(s_j)$ is *weakly* better than any higher bid for bidder j with signal s_j. Note that we do not claim a strict payoff ranking because in profile η bidder j (unlike bidder k) might win with zero probability at both bids. □

Recall that we have defined $v_{min}(s_i) = \min_{s_{-i} \in \{S_{-i}} v_i(s_i, s_{-i})\}$ and observe that $\mu_i(s_i) = v_{min}(s_i)$. We next observe that bidder i with signal s_i that only submits undominated bids never bids below $\mu_i(s_i)$.

Observation 30. *In the original game λ and in any tremble $\lambda(\epsilon, R)$ for $\epsilon > 0$, for bidder i with signal s_i bidding $\mu_i(s_i)$ weakly dominates bidding any amount $b_i < \mu_i(s_i)$.*[10]

We combine this observation with Lemma 29 to the prove Theorem 11.

Proof. (of Theorem 11) Fix any strict linear order on the signals that is consistent with the order of lower bounds they place on the expected value. That is, fix an arbitrary order satisfying that for every s_i and s_j, if $\mu_i(s_i) > \mu_j(s_j)$ then s_i is ranked higher than s_j.

The proof proceeds by induction. The base case considers the highest signal according to the fixed order. Suppose the highest signal is bidder i's signal s_i. By Observation 30 bidding $\mu_i(s_i) = v_{min}(s_i)$ dominates any lower bid for bidder i with signal s_i. Moreover, SHSP implies that for the highest signal s_i, for any $s_{-i} \in S_{-i}$ such that (s_i, s_{-i}) is feasible, it holds that $\mu_i(s_i) = v(s_i, s_{-i})$. Thus $\mu_i(s_i) = v_{max}(s_i)$ and therefore in any tremble $\lambda(\epsilon, R)$ in which the bid of agent i with signal s_i belongs to $[v_{min}(s_i), v_{max}(s_i)]$ it holds that the bid must be $\mu_i(s_i) = v_{min}(s_i) = v_{max}(s_i)$. Moreover, bidding $\mu_i(s_i)$ is a dominant strategy for bidder i with signal s_i in the original game λ and any tremble $\lambda(\epsilon, R)$.

We move to the induction step. Consider the l^{th} highest signal, which is s_j received by bidder j. Assume that every bidder i with strictly higher signal s_i (that is, $\mu_i(s_i) > \mu_j(s_j)$) bids $\mu_i(s_i)$ with probability 1. Observation 30 and claim (2) of Lemma 29 imply that it is a best response for bidder j with signal s_j to bid $\mu_j(s_j)$ in the original game λ and the tremble $\lambda(\epsilon, R)$. Moreover, Observation 30 and claim (1) of Lemma 29 imply that this is the *unique* best response in any NE in undominated bids of any tremble $\lambda(\epsilon, R)$, for $\epsilon > 0$.

Proceeding by induction through all signals shows that the pure strategy profile μ is a Nash equilibrium both in the original game λ and in any tremble $\lambda(\epsilon, R)$ with $\epsilon > 0$. Moreover, it is the unique Nash equilibrium in undominated bids in any tremble $\lambda(\epsilon, R)$ with $\epsilon > 0$. The theorem follows directly. □

C.2 Relation to the work of Einy et al.(2002)

Einy et al. (2002) study common value second price auction in domains that are *connected*. For *connected domains* Einy et al. consider the concept of *sophisticated equilibrium*, which makes successive rounds of dominated strategy eliminations. This process might result in multiple equilibria and that paper points out a single sophisticated equilibrium that Pareto-dominates the rest in terms of bidders resulting utilities, and it is also the only sophisticated equilibrium that guarantees every bidder non-negative utility. Moreover, this is the only sophisticated equilibrium that survives the elimination process if an uninformed bidder is added to the domain.

In this section we observe that Theorem 11 applies to any connected domain, as any such domain satisfies the strong-high-signal property. Moreover, we observe that for connected domains the TRE of Theorem 11 is exactly the one pointed out by Einy et al. (2002). Finally, we show that some domain that satisfy the strong-high-signal property are not connected. Some obvious such domains are monotonic domains in which the mapping from the state of the world to signals is not deterministic (yet they still satisfy the strong-high-signal property), but we also present examples of domains in which the mapping is deterministic yet they are not connected and for which Theorem 11 applies.

Before formally presenting connected domains we present an example due to Einy et al. (2002) and the TRE we (as well as Einy et al.) pick for that domain.

[10] It is trivial to come up with strategies for the other bidders for which $\mu_i(s_i)$ gives strictly higher utility than b_i.

Example 31. *Assume that there are two buyers and four states of the world* $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$, *with* $v(\omega_i) = i$ *and states all are equally probable (* $H(\omega_i) = 1/4$ *for all* $i \in \{1, 2, 3, 4\}$ *). If the state is* ω_1 *then agent 1 gets the signal* L_1*, otherwise he gets* H_1*. If the state is* ω_4 *then agent 2 gets the signal* H_2*, otherwise he gets* L_2*. In* μ *,the TRE of Theorem 11 it holds that* $\mu_2(H_2) = v(H_1, H_2) = 4$, $\mu_1(H_1) = v(H_1, L_2) = 2.5$ *and* $\mu_1(L_1) = \mu_2(L_2) = v(L_1, L_2) = 1$.

We next define connected domains.

Definition 32. *A domain is a* connected domain *if the following hold. Each agent* i *has a partition* Π_i *of the state of nature and his signal is the element of the partition that include the realized state. The information partition* Π_i *of bidder* i *is* connected *(with respect to the common value* v*) if every* $\pi_i \in \Pi_i$ *has the property that, when* $\omega_1, \omega_2 \in \pi_i$ *and* $v(\omega_1) \leq v(\omega_2)$ *then every* $\omega \in \Omega$ *with* $v(\omega_1) \leq v(\omega) \leq v(\omega_2)$ *is necessarily in* π_i*. A common-value domain is* connected *(with respect to the common value) if for every agent* i *his information partition* Π_i *is connected.*

Lemma 33. *Every connected domain satisfies that strong-high-signal property.*

Proof. Let Π^* be the coarsest partition of Ω that refines the partition Π_j for every agent j. Let σ denote an element of Π^*. Let $v(\sigma)$ denote the expected value of the item conditional on σ. We prove the claim by induction on the number of elements in Π^*. If this number is 1 the claim trivially holds as the domain in which no agent gets any information satisfies the property by definition.

Assume that we have proven the claim for every Π^* of size smaller than k, we prove the claim for Π^* of size k. Consider that element σ of Π^* such that $v(\sigma)$ is maximal. There must exist an agent i and signal s_i such that s_i implies σ, otherwise Π^* is not the coarsest refinement. There is only one combination of signals that has value $v(\sigma)$, in that combination each agent gets the best signal (the one with the highest value conditional on the signal). Now, as the domain is connected it holds that $v(\sigma) > v(t)$ for every combination of signals t. This implies that s_i has the required properties from the top signal at a domain that satisfies the strong-high-signal property. Removing this signal creates another connected domain, and its coarsest partition has only $k - 1$ elements, so by the induction hypothesis it satisfies the strong-high-signal property. We conclude that the original domain satisfies the strong-high-signal property as we need to show. □

Proposition 34. *For every connected domain the TRE of Theorem 11 is exactly the same as the unique sophisticated equilibrium picked by Einy et al. (2002) (the sophisticated equilibrium that survives the elimination process if an uninformed bidder is added to the domain).*

Proof. Einy et al. show that unique sophisticated equilibrium that they pick can be computed as follows. One can look at Π^*, the coarsest partition of Ω that refines the partition Π_j for every agent j. Let σ denote an element of Π^*. Let $v(\sigma)$ denote the expected value of the item conditional on σ. An order over elements $\sigma_1, \sigma_2 \in \Pi^*$ is naturally defined by the order on the corresponding values $v(\sigma_1)$ and $v(\sigma_2)$. For agent j with signal $\pi_j \in \Pi_j$ the bid is defined to $\min_{\sigma \in \pi_j} v(\sigma)$. An equivalent definition is that agent j with signal $\pi_j \in \Pi_j$ bids $min\{v(\pi_j, \pi_{-j}) | \pi_{-j} \in S_{-j} \ and \ (\pi_j, \pi_{-j}) \ is \ feasible\}$, which is exactly $\mu_j(s_j)$ as defined in Theorem 11. □

We next show that there are domains that are not connected yet satisfy the strong-high-signal property. This implies that Theorem 11 applies to a strict superset of the domains that are handled by Einy et al. (2002). We start with a simple example with only one informed bidder.

Example 35. *Consider a domain with two buyers and three states of the world* $\Omega = \{\omega_1, \omega_2, \omega_3\}$*, with* $v(\omega_1) = 0$*,* $v(\omega_2) = 4$*,* $v(\omega_3) = 10$ *and all states are equally probable (*$H(\omega_i) = 1/3$ *for all* $i \in \{1, 2, 3\}$*). If the state is* ω_1 *or* ω_3 *then agent* 1 *gets the signal* H_1*, otherwise he gets* L_1*. Agent* 2 *is not informed at all. This example is covered by Theorem 11 and moreover it is covered by Theorem 5. Yet, this domain is not connected, as signal* H_1 *of agent* 1 *indicates that the state is* ω_1 *or* ω_3 *and does not include* ω_2*.*

We also present an example with more than one informed bidder, in this example there are 2 agents and each has a binary signal.

Example 36. *Assume that there are two buyers and four states of the world* $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ *with* $v(\omega_1) = 0$*,* $v(\omega_2) = 4$*,* $v(\omega_3) = 6$*,* $v(\omega_4) = 10$*, and all states are equally probable (*$H(\omega_i) = 1/4$ *for all* $i \in \{1, 2, 3, 4\}$*). If the state is* ω_4 *then agent 1 gets the signal* H_1*, otherwise he gets* L_1*. If the state is* ω_1 *or* ω_3 *then agent 2 gets the signal* L_2*, otherwise he gets* H_2*. (note that this is not connected as* ω_2 *does not belong to* L_2*). In the TRE* μ *of Theorem 11 it holds that* $\mu_1(H_1) = v(H_1, H_2) = 10$*,* $\mu_2(H_2) = v(L_1, H_2) = 4$ *and* $\mu_1(L_1) = \mu_2(L_2) = v(L_1, L_2) = 3$*.*

While Example 35 presents a very simple domain that is not connected, it is clear that there exists a different representation of the states of the world for which a domain with exactly the same signal structure and posteriors, is indeed connected. In this new representation each state corresponds to one of the informed agent's signals and the value corresponds to the posterior value given that signal. That is, we can define $\Omega' = \{\omega'_1, \omega'_2\}$, with $v(\omega'_1) = 5$, $v(\omega'_2) = 4$, and the probabilities are $H(\omega'_1) = 2/3$ and $H(\omega'_2) = 1/3$. If the state is ω'_1 then agent 1 gets the signal H_1, otherwise he gets L_1. Agent 2 is not informed at all. Clearly under the new representation the domain is connected, and the domain is equivalent to the original domain.

One might wonder if *any* domain that satisfies the strong-high-signal property can be transformed to an equivalent connect domain. We next show that this is not the case, presenting a domain that satisfies the property and cannot be represented by a connect domain. This shows that Theorem 11 applies to domains that do not have a representation as connected domains.

The domain we consider is the domain presented in Example 36, with $v(\omega_2)$ assigned a value of 2 instead of 4. Clearly in a connected domain that is equivalent to that domain it must be the case that signals H_1 and H_2 are both received for some subset of states of the world such that for each such state the value is at least as high as the value if signal H_1 is not received. Now connectivity for H_2 implies that $v(L_1) \geq v(L_2)$ which does not hold for the domain we are considering.

C.3 Generalizing "Lemons and Peaches" to n agents

We restate Proposition 15 and present its proof.

Proposition 37. *Fix any non-negative constants* $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$*. Consider any monotonic domain that satisfies the strong-high-signal property and for which it also holds that every agent* $i \in \{1, 2, \ldots, n\}$ *is* ϵ_i*-informed about peaches. In the unique TRE* μ *(as defined in Theorem 11) the revenue of the SPA is at least*

$$E[v(\omega)] - \sum_{j=1}^{n} \epsilon_j$$

Proof. If $\sum_{i=1}^{n} \epsilon_i \geq 1$ the claim follows trivially. Items have values in $[0,1]$ and thus $E[v(\omega)] \leq 1$, this implies that $E[v(\omega)] - \sum_{j=1}^{n} \epsilon_j \leq 0$, and the claim about the revenue clearly holds as revenue is non-negative since every bid is non-negative. We next assume that $\sum_{i=1}^{n} \epsilon_i < 1$.

Let $L = (L_1, L_2, \dots, L_n)$ be the combination of signals in which each agent i gets signals L_i. Observe that $Pr[not\ L] \leq \sum_{i=1}^{n} Pr[s_i \neq L_i] \leq \sum_{i=1}^{n} \epsilon_i$ as every agent i is ϵ_i-informed about peaches, thus $Pr[L] \geq 1 - \sum_{j=1}^{n} \epsilon_j > 0$ which means that L is feasible. As the domain is monotonic and L_i is the lowest signal for agent i, for every feasible s it holds that $v(L) \leq v(s)$. This implies that $\mu_i(s_i) \geq v(L)$ for every agent i and signal $s_i \in S_i$.

As all bids are at least $v(L)$, the revenue is at least $v(L)$, thus it is sufficient to show that $v(L) \geq E[v(\omega)] - \sum_{j=1}^{n} \epsilon_j$.

Observe that

$$E[v(\omega)] = v(L) \cdot Pr[L] + v(not\ L) \cdot Pr[not\ L]$$

Which implies that

$$v(L) = \frac{E[v(\omega)] - v(not\ L) \cdot Pr[not\ L]}{Pr[L]} \geq E[v(\omega)] - Pr[not\ L] \geq E[v(\omega)] - \sum_{i=1}^{n} \epsilon_i$$

since $0 < Pr[L] \leq 1$, $v(not\ L) \leq 1$ (as for any ω it holds that $v(\omega) \in [0,1]$), and $Pr[not\ L] < \sum_{i=1}^{n} \epsilon_i$. □

We next restate Proposition 17 and present its proof.

Proposition 38. *Fix any positive constants $\epsilon_1, \epsilon_2, \dots, \epsilon_i$. Consider any domain that satisfies the strong-high-signal property and for which it also holds that each agent $j \in \{1, 2, \dots, i-1\}$ is ϵ_j-informed about peaches, while agent i is ϵ_i-informed about lemons. Assume that the domain is non degenerated in the following sense:*

- *For any $j < i$ the signal H_i does not imply L_j (alternatively, $(L_j, s_i, s_{-\{i,j\}})$ is feasible for some $s_i \neq H_i$ and some $s_{-\{i,j\}}$).*
- *For any $j > i$ and any signal $s_j \in S_j$, the signal H_i does not imply s_j (alternatively, $(s_j, s_i, s_{-\{i,j\}})$ is feasible for some $s_i \neq H_i$ and some $s_{-\{i,j\}}$).*

Then the revenue of the SPA in the unique TRE μ (as defined in Theorem 11), is at most

$$\epsilon_i + \sum_{j=1}^{i} \epsilon_j$$

Proof. If $\sum_{i=1}^{n} \epsilon_i \geq 1$ the claim follows trivially. Items have values in $[0,1]$ and thus all bids are at most 1, which implies that the revenue is at most 1. We next assume that $\sum_{i=1}^{n} \epsilon_i < 1$.

Since each $j < i$ is ϵ_j-informed about peaches it holds that

$$Pr[L_1, L_2, \dots, L_{i-1}] \geq 1 - \sum_{j=1}^{i-1} \epsilon_j$$

Now, since i is ϵ_i-informed about lemons it holds that $Pr[H_i] \geq 1 - \epsilon_i$, and thus

$$Pr[L_1, L_2, \dots, L_{i-1}, H_i] \geq Pr[L_1, L_2, \dots, L_{i-1}] + Pr[H_i] - 1 \geq 1 - \sum_{j=1}^{i} \epsilon_j > 0$$

The revenue obtained when the signals of agents $1, 2, \ldots, i$ are *not* realized to $(L_1, L_2, \ldots, L_{i-1}, H_i)$ is at most the maximal value of any item, which is 1, and that happens with probability at most $\sum_{j=1}^{i} \epsilon_j$. Thus this case contributes at most $\sum_{j=1}^{i} \epsilon_j$ to the expected revenue.

We next bound the revenue obtained when the signals of agents $1, 2, \ldots, i$ are realized to $(L_1, L_2, \ldots, L_{i-1}, H_i)$, and event that happens with probability at most 1. To prove the claim it is sufficient to show that the maximal of the bids of all agents other than i is at most ϵ_i, since this upper bounds the revenue. We first bound the bid $\mu_j(L_j)$ of any agent $j < i$ when getting signal L_j. By the first non-degeneracy assumption $(L_j, s_i, s_{-\{i,j\}})$ is feasible for some $s_i \neq H_i$ and some $s_{-\{i,j\}}$. As agent i is ϵ_i-informed about lemons it holds that $v(L_j, s_i, s_{-\{i,j\}}) \leq \epsilon_i$ and thus $\mu_j(L_j) \leq \epsilon_i$.

We next bound the bid $\mu_j(s_j)$ of any agent $j > i$ when getting any signal $s_j \in S_j$. By the second non-degeneracy assumption $(s_j, s_i, s_{-\{i,j\}})$ is feasible for some $s_i \neq H_i$ and some $s_{-\{i,j\}}$. As agent i is ϵ_i-informed about lemons it holds that $v(s_j, s_i, s_{-\{i,j\}}) \leq \epsilon_i$ and thus $\mu_j(s_j) \leq \epsilon_i$. We have shown that when the signals of agents $1, 2, \ldots, i$ are realized to $(L_1, L_2, \ldots, L_{i-1}, H_i)$ the maximal of the bids of all agents other than i is at most ϵ_i, thus the revenue in this case is bounded by ϵ_i, and the claim follows. □

D Two Agents, Each with a Binary Signal

D.1 Proof of Lemma 20

Let G be a distribution function. We say that G has an *atom* at b if $b = 0$ and $G(0) > 0$, or if $b > 0$ and G is discontinuous at b. We define $G^-(b) = \sup_{x<b} G(x)$. We say that a bid b of bidder j is *optimal* (or is in the *support*) if the utility from that bid (given the other agent's strategy and the random bidder) is at least as high as with any other bid.

In this section we use i to denote a bidder, either bidder 1 or 2. When we want to refer to the other bidder we use j to denote that bidder, and assume that $j \neq i$.

Let R be a standard distribution and fix some $\epsilon > 0$. A NE η of the (ϵ, R)-tremble of the game λ consists of four bid distributions, one for each bidder for each signal he may receive. Thus $\eta = (G_1, G_1^L, G_2, G_2^L)$ where G_i and G_i^L are the bid distributions when $i \in \{1, 2\}$ gets the signals H_i and L_i, respectively. In η, for bidder i with signal H_i: let $\Pi_i(b_i)$ denote the utility (profit) of bidder i when he bids b_i, and let $v_i^{win}(b_i)$ denote the expected value of the items i gets, conditional on winning, when he bids b_i.

To simplify the notation we denote $v_1 = v(H_1, L_2)$ and $v_2 = v(L_1, H_2)$. We assume that $0 < Pr[H_1, L_2](1 - v_1) \leq Pr[L_1, H_2](1 - v_2) < 1$, and that in case of equality $v_1 \geq v_2$. Note that this implies that $\min\{Pr[H_1, L_2], Pr[L_1, H_2]\} > 0$.

We first show that if bidders never submit dominated bids bidder $i \in \{1, 2\}$ that receives signal L_i must bid $v(L_1, L_2) = 0$.

Lemma 39. *At η the following must hold. For each bidder $i \in \{1, 2\}$ it holds that $G_i^L(0) = 1$. That is, bidder i with signal L_i always bids $v(L_1, L_2) = 0$. Also, it holds that $G_i^-(v_i) = sup_{b<v_i} G_i(b) = 0$. That is, bidder i with signal H_i always bids at least v_i.*

Proof. By assumption, bidders do not make weakly dominated bids. Therefore, bidder i bids at least 0 given signal L_i and at least $v(H_i, L_j)$ given signal H_i. Similarly, bidder i bids no more than $v(L_i, H_j)$ given signal L_i and no more than 1 given signal H_i. Bidder 1 with signal L_1 cannot bid $b \in (0, v_2)$ because she would only win when bidder 2 has a low signal and the value is zero but she would pay a positive amount due to the random bidder. Increasing the bid to v_2 incurs the same losses conditional on L_2 as bidding just below v_2 and earns zero conditional on H_2 because any wins are priced at their value v_2. Therefore bidder 1 must bid 0 given a low signal, and the same is true for bidder 2 by similar logic. $\square$

Given this lemma we focus in the rest of the proof at the bidding of each bidder i given his high signal H_i. (e.g. if we say that some bid "is optimal for i" we mean to say that this bid "is optimal for i with signal H_i").

Lemma 40. *At η the following must hold. Assume that G_j is discontinuous at $b < 1$ (has an atom at b), then $\exists \delta > 0$ such that bidding in the interval $(b - \delta, b]$ is not optimal for i as it is dominated by bidding $b + \delta$.*

Proof. Let Δ be the discrete increase in G_j at b. For $\delta > 0$ small enough bidding $b + \delta$ is strictly better than bidding in $(b - \delta, b]$ as the probability of winning increases by at least $\Delta/2$ (moving from b to $b + \delta$ means always winning against the atom instead of tie-breaking), while the increase in payment when winning low value items tends to 0 as δ go to zero (as the random bidder is bidding continuously). $\square$

Let b^- and b^+ be two bids such that $0 \leq b^- < b^+ \leq 1$.

For $b > 0$ define $G_j^-(b)$ as the left hand limit of G_j evaluated at b:

$$G_j^-(b) = \sup_{x<b} G_j(x).$$

Lemma 41. *At η the following must hold. For every bidder j the expected value of the items he gets, conditional on winning, is monotonic in his bid. That is, if $v_j^{win}(b)$ is the expected value of the items j gets, conditional on winning, with bid b, then $v_j^{win}(b)$ is non-decreasing in b.*

Proof. If i is bidding an atom of size $\Delta_i(b)$ at b it holds that $\Delta_i(b) = G_i(b) - G_i^-(b)$. If bidder j is bidding $b > 0$, then j's expected value conditional on winning $v_j^{win}(b)$ can be computed by separating the case that i bids below b, and the case that i is bidding at b:

$$v_j^{win}(b) = \frac{\Pr[H_i \mid H_j]\left(G_i^-(b) + \frac{1}{2}\Delta_i(b)\right) + \Pr[L_i \mid H_j] v_j}{\Pr[H_i \mid H_j]\left(G_i^-(b) + \frac{1}{2}\Delta_i(b)\right) + \Pr[L_i \mid H_j]}$$

where the factor half comes from tie breaking in case both are bidding at b. Note that the first term is non-decreasing in b because it is increasing in $G_i^-(b)$ and $G_i^-(b)$ is non-decreasing in b. Moreover, any increase in the bid will make sure the bidder always wins against the atom at b, instead of only half of the time. $\square$

The next lemma shows than an optimal bid b for bidder j must be at least the expected value of the item j wins, conditional on winning.

Lemma 42. *At η the following must hold. If $b \in [0, 1)$ is an optimal bid of bidder j then $b \geq v_j^{win}(b)$.*

Proof. If $v_j^{win}(b) = v_j$ the claim follows from $b \geq v_j$ (Lemma 39).

Now assume in contradiction that $b < v_j^{win}(b) \leq 1$ and that $v_j^{win}(b) > v_j$. It must hold that $G_i(b) > 0$, since $G_i(b) = 0$ implies $v_j^{win}(b) = v_j$. If i has an atom at $b < 1$ then b is not optimal for j by Lemma 40, contradicting our assumption that b is optimal for j. Thus, bidder i does not have an atom at b.

We show that for $\delta > 0$ that is small enough ($\delta < v_j^{win}(b) - b$), bidding $b + \delta$ gives higher utility. To show that the bid $b + \delta$ gives higher utility than b, we consider the difference in utility due to such an increase in the bid. There are two cases: first, if bidder j wins with $b + \delta$ but would have lost with b due to a bid of i in $(b, b + \delta)$ then j wins an item of value 1 and pays at most $b + \delta < 1$, having positive utility. Second, if i was not bidding in $(b, b + \delta)$ but the random bidders does, by bidding $b + \delta$ bidder j is now winning items with expected value at least $v_j^{win}(b)$ (by Lemma 41 $v_j^{win}()$ is non-decreasing) and paying at most $b + \delta$. As $\delta < v_j^{win}(b) - b$ the expected value from such a win is positive. Moreover, this second event happens with strictly positive probability because the random bidder is bidding continuously over $[0, 1]$ and given no atoms of bidder i at $b > 0$ bidder i bids less than b with probability $Pr[L_i|H_j] + Pr[H_i|H_j]G_i(b) > 0$. We conclude that such an increase in bid strictly increases the utility. □

Lemma 43. *At η the following must hold. For $1 \geq b^+ > b^- \geq 0$ suppose that $G_j(b^-) = G_j^-(b^+)$ (j does not bid on (b^-, b^+)). Let $\Gamma = G_j(b^-)$. Let*

$$b_i^*(\Gamma) = \frac{\Pr[H_j|H_i]\,\Gamma + \Pr[L_j|H_i]\,v_i}{\Pr[H_j|H_i]\,\Gamma + \Pr[L_j|H_i]}$$

If $b_i^*(\Gamma) \in (b^-, b^+]$ then $b_i^*(\Gamma)$ strictly dominates any other bid by i in $(b^-, b^+]$. If $b_i^*(\Gamma) > b^+$, then i's payoff is strictly increasing in b over $(b^-, b^+]$. If $b_i^*(\Gamma) \leq b^-$, then i's payoff is strictly decreasing in b over (b^-, b^+).

Proof. $G_j(b_j)$ is constant over (b^-, b^+) and thus $g_j(b_j) = 0$ over (b^-, b^+). Therefore $\Pi_i(b_i)$ is continuous and differentiable on (b^-, b^+). Moreover, since $g_j(b_j)$ is zero, the derivative is

$$\frac{d\Pi_i(b_i)}{db_i} = \hat{r}(b_i)\big(\Pr[H_j|H_i]\,\Gamma\,(1 - b_i) + \Pr[L_j|H_i]\,(v_i - b_i)\big)$$

As we assume that $1 > \Pr[L_j|H_i] > 0$ and it holds that $\hat{r}(b_i) > 0$, $\Pr[H_j|H_i] \geq 0$, this function of b_i is not identically 0. The function has a unique 0 at $b^*(\Gamma)$, it is positive for $b_i > b^*(\Gamma)$, and it is negative for $b_i > b^*(\Gamma)$. Thus if i has an optimal bid in the interval (b^-, b^+) it can only be at $b^*(\Gamma)$. To extend the result to the interval $(b^-, b^+]$ consider four cases:

(1) $b^*(\Gamma) \in (b^-, b^+)$: In this case, $b^*(\Gamma)$ is the unique best bid within the interval (b^-, b^+). At either endpoint b^- or b^+, either j bids an atom and the corresponding endpoint cannot be an optimal bid (Lemma 40) or j does not bid an atom and $\Pi_i(b_i)$ is continuous at that point, meaning it is dominated by the interior bid $b_i^*(\Gamma)$. In either case, $b_i^*(\Gamma)$ is the only possible optimal bid within $[b^-, b^+]$.

(2) $b^*(\Gamma) \leq b^-$: In this case, $d\Pi_i(b_i)/db_i < 0$ for (b^-, b^+) and there is no optimal bid in (b^-, b^+). If j bids an atom at b^+ then b^+ is not an optimal bid for i by Lemma 40. If j does not bid an atom at b^+, then $\Pi_i(b_i)$ is continuous from the left at b^+. Therefore $\Pi_i(b_i)$ is strictly lower at b^+ than at any other $b_i \in (b^-, b^+)$. In either case there is no optimal bid within $(b^-, b^+]$, a contradiction.

(3) $b^*(\Gamma) > b^+$: Lemmas 41 and 42 imply that any optimal bid $b_i > b^-$ must be at least $b_i^*(\Gamma)$ because $v_i^{win}(b_i) = b_i^*(\Gamma)$ for all $b_i \in (b^-, b^+)$. Therefore there is no optimal bid at or below b^+, a contradiction.

(4) $b^*(\Gamma) = b^+$. In this case, $d\Pi_i(b_i)/db_i > 0$ for (b^-, b^+). Therefore $\Pi_i(b_i)$ is strictly higher at b^+ than at any $b_i \in (b^-, b^+)$ because $\Pi_i(b_i)$ is either continuous at b^+ or increases discretely at b^+ (depending on whether or not j has an atom at b^+.) Therefore $b^*(\Gamma) = b^+$ is the only possible optimal bid for i in the interval $(b^-, b^+]$.

□

Corollary 44. *At η the following must hold. If bidder $i \in \{1,2\}$ bids an atom at $b \in [0,1]$, then $b = b_i^*(G_j(b))$.*

Proof. Lemma 40 implies that j does not bid in the interval $(b-\delta, b]$ for some $\delta > 0$. Therefore Lemma 43 implies the result. □

Define $\underline{b}_i$ to be the infimum bid by $i \in \{1,2\}$, $\underline{b}_i = \inf\{b : G_i(b) > 0\}$. Let $\underline{b} = min\{\underline{b}_1, \underline{b}_2\}$ be the infimum of all bids of any bidder with a high signal. Let $b_{min} = \max\{\underline{b}_1, \underline{b}_2\}$.

Corollary 45. *At η the following must hold. Suppose that $j \in \{1,2\}$ has an optimal bid b at or below $\underline{b}_i$. Then $b = v_j$.*

Proof. Note that $G_i(b) = 0$ because Lemma 40 implies that i does not have an atom at $\underline{b}_i$ if $b = \underline{b}_i$. Thus, as i does not bid strictly below $\underline{b}_i$ but j has an optimal bid b weakly below $\underline{b}_i$, Lemma 43 implies $b = b_j^*(0) = v_j$.

□

Lemma 46. *At η the following must hold. Assume that both bids $b^- \geq 0$ and $b^+ > b^-$ are optimal bids for bidder $i \in \{1,2\}$. Then for bidder $j \neq i$ it holds that $G_j(b^+) > G_j(b^-)$.*

Proof. Assume in contradiction that both bids b^+ and $b^- < b^+$ are optimal bids for bidder i, while $G_j(b^+) = G_j(b^-) = \Gamma$ (note that G_j is non decreasing thus $G_j(b^+) \geq G_j(b^-)$). Because $G_j(b)$ is constant over $[b^-, b^+]$, $\Pi_i(b_i)$ is continuous and differentiable on (b^-, b^+) and is continuous from the left at b^+. By Lemma 40, the fact that b^- is an optimal bid for i implies that j does not bid an atom at b^-. Therefore there is no tie-breaking at b^- that would be resolved by bidding slightly more than b^- and $\Pi_i(b_i)$ is also continuous from the right at b^-. Following the argument in the proof of Lemma 43, within the interval (b^-, b^+), $d\Pi_i(b_i)/db_i$ is zero at $b^*(\Gamma)$, strictly positive for $b_i > b^*(\Gamma)$, and strictly negative for $b_i > b^*(\Gamma)$. There are three cases: (1) $b^*(\Gamma) \leq b^-$. Then $d\Pi_i(b_i)/db_i < 0$ for (b^-, b^+) and, by right-continuity at b^- and left-continuity at b^+, $\Pi_i(b^+) < \Pi_i(b^-)$ contradicting optimality of b^+. (2) $b^*(\Gamma) \geq b^+$. Then by a symmetric argument b^- cannot be optimal. (3) $b^*(\Gamma) \in (b^-, b^+)$. Then by similar argument both b^- and b^+ are strictly dominated by $b^*(\Gamma)$. □

Lemma 47. *At η the following must hold.*

1. *Suppose both bidders have the same infimum bid: $\underline{b}_i = \underline{b}_j = \underline{b} = b_{min}$. Then $\underline{b} = \max\{v_i, v_j\}$. If $v_i = v_j$, then neither bidder bids an atom at $\underline{b}$ (that is, $G_j(\underline{b}) = G_i(\underline{b}) = 0$). However, if $v_i < v_j$ then j bids an atom at $\underline{b} = v_j$ and i does not bid at $\underline{b}$.*

2. *Suppose bidder i has a strictly higher infimum bid: $\underline{b}_i > \underline{b}_j$. Then $\underline{b} = \underline{b}_j = v_j$ and j bids an atom with some positive weight $\Gamma > 0$ at v_j but nowhere else at or below $\underline{b}_i$:*

$$G_j(b) = \begin{cases} 0 & b < v_j \\ \Gamma & b \in [v_j, \underline{b}_i] \end{cases}$$

Moreover, $b_{min} = \underline{b}_i > v_i$.

Proof. (1) It cannot be the case that both bidders have an atom at $\underline{b}$. Suppose i does not have an atom at $\underline{b}$. Then $\Pi_j(b)$ is continuous at $\underline{b}$ and therefore $\underline{b}$ is an optimal bid for j. ($\underline{b}_j = \underline{b}$ implies that j bids with positive probability at $\underline{b}$ or in every neighborhood above $\underline{b}$.) Because j has an optimal bid at $\underline{b}_i$, Corollary 45 implies that $\underline{b}_i = v_j$. Moreover, $\underline{b}_i \geq v_i$ by Lemma 39. Therefore $v_i \leq v_j$ and $\underline{b} = \max\{v_i, v_j\}$.

Suppose that $v_i < v_j$ and j does not bid an atom at $\underline{b}$. Then $\Pi_i(b)$ is continuous at $\underline{b}$ and hence $\underline{b}$ is an optimal bid for i and Corollary 45 implies $\underline{b} = v_i$, which is a contradiction. Thus $v_i < v_j$ implies j has an atom at $\underline{b}$. (Hence by Lemma 40 i does not bid at $\underline{b}$.)

Suppose that $v_i = v_j$ and j has an atom of weight $\Gamma > 0$ at $\underline{b}$. Then by Lemma 42, bidder i must bid at least $v_i^{win}(\underline{b}) > v_i$, which is a contradiction. Thus $v_i = v_j$ implies neither bidder has an atom at $\underline{b}$.

(2) The assumption $\underline{b}_j < \underline{b}_i$ implies that j bids with some positive probability $\Gamma > 0$ below $\underline{b}_i$. By Corollary 45, j can only bid below $\underline{b}_i$ at v_j. Therefore j bids with atom Γ at $\underline{b}_j = v_j$ and nowhere else below $\underline{b}_i$. Moreover, Lemma 42 implies that for all bids $b \geq \underline{b}_i$, bidder i must bid at least $v_i^{win}(\underline{b}_i) = b_i^*(\Gamma) > v_i$.

□

Lemma 48. *If for all $\delta > 0$, bidder i has an optimal bid in the interval $(b - \delta, b]$ then b is an optimal bid for i.*

Proof. By Lemma 40, j does not have an atom at b and hence $\Pi_i(b_i)$ is continuous from the left at $b_i = b$. Since i has an optimal bid at b or arbitrarily close to b, continuity implies that b must be an optimal bid. □

Suppose that bidder j has an atom at $b > 0$. By Lemma 40, bidder i does not bid in $(b - \delta, b]$ for some $\delta > 0$. Define $x_i(b)$ to be the supremum point below b at which bidder i does place a bid

$$x_i(b) = \sup\{x : G_i(x) < G_i(b)\} = \inf\{x : G_i(x) = G_i(b)\}.$$

Lemma 40 implies $x_i(b) < b$. Similarly, let

$$x_j(b) = \inf\left\{x : G_j(x) = G_j^-(b)\right\}.$$

Note that if i does not bid below b ($\underline{b}_i \geq b$) then $x_i(b) = -\infty$.

Our goal is to prove that if j has an atom at b then b is j's infimum bid. We first prove some helpful claims.

Lemma 49. *If j has an atom at $b > 0$ and b is not j's infimum bid ($0 \leq \underline{b}_j < b$) then:*

1. *It holds that $v_j \leq x_j(b) < x_i(b) < b$.*
2. *In the interval $(x_j(b), b]$, i bids an atom at $x_i(b) = b_i^*(G_j(x_i(b)))$ but nowhere else.*
3. *j bids with an atom at $x_j(b) = b_j^*(G_i(x_j(b)))$.*
4. *$b = b_j^*(G_i(b))$.*

Proof. We prove the claims:

1. We prove that $v_j \leq x_j(b) < x_i(b) < b$:
   - $x_j(b) \geq v_j$: By assumption ($\underline{b}_j < b$) bidder j bids with positive probability below b. Such bids must be at least v_j.
   - $x_i(b) < b$: follows from Lemma 40.
   - $x_j(b) < x_i(b)$: suppose not and $x_i(b) \leq x_j(b) < b$. There are two cases. (i) First, if $x_j(b) > x_i(b)$, then there exists some bid $b^- \in [x_i(b), b)$ where j bids. Then by Lemma 46, $G_i(b^-) < G_i(b)$ which contradicts $G_i(x_i(b)) = G_i(b)$ and $x_i(b) < b^- < b$. (ii) Second, if $x_j(b) = x_i(b)$ then by Lemma 48 $x_i(b)$ is an optimal bid for j. Then by Lemma 46, $G_i(x_i) < G_i(b)$ which contradicts $G_i(x_i) = G_i(b)$.

2. By part (1) and definition of $x_i(b)$, j does not bid with positive probability in the interval $(x_j(b), b)$ but i does. As a result, Lemma 43 implies part (2).
3. There are two cases, either $\underline{b}_i = x_i(b)$ or $\underline{b}_i < x_i(b)$. (i) By part (1), j bids with positive probability below $x_i(b)$. Therefore, if bidder i's infimum bid is at $\underline{b}_i = x_i(b)$, Lemma 47 implies that j bids with an atom at $x_j(b) = v_j$. (ii) bidder i bids with positive probability below $x_i(b)$ and $\underline{b}_i < x_i(b)$. Parts (1) and (2) of the Lemma can be applied to the atom at $x_i(b)$ and these imply that j bids with an atom at $x_j(b) = b_j^*(G_i(x_j(b))) > v_i$.
4. Since i does not have an atom at b (Lemma 40), $v_j^{win}(b) = b_j^*(G_i(b))$. Therefore Lemma 42 implies that $b \geq b_j^*(G_i(b))$. Thus it is sufficient to show $b \leq b_j^*(G_i(b))$. Suppose not and $b > b_j^*(G_i(b))$. Within the interval $(x_i(b), b)$, the proof of Lemma 43 implies Π_j is strictly increasing as the bid is moved towards $b_j^*(G_i(b))$ from above or below. Since i does not have an atom at b, Π_j is left-continuous at b. Thus if $b > b_j^*(G_i(b))$, b could not be optimal for j since it would be dominated by bidding $b - \delta$ for some $\delta > 0$.

□

Lemma 50. *If $j \in \{1, 2\}$ has an atom at b then b is j's infimum bid: $b = \underline{b}_j$.*

Proof. Suppose not and j bids with positive probability in a neighborhood of $b^- < b$. Then by Lemma 49, j bids with an atom at $x_j(b) = b_j^*(G_i(x_j(b)))$, i bids with an atom at $x_i(b) \in (x_j(b), b)$, $b = b_j^*(G_i(b))$, and there are no other bids in the interval $(x_j(b), b)$. We will show a contradiction by showing that $\Pi_j(b) > \Pi_j(x_j(b))$. Let $\Gamma_1 = G_i(x_j(b))$ and $\Gamma_2 = G_i(x_i(b)) = G_i(b)$.

Let Π_j^- and Π_j^+ be the left and right hand limits of Π_j respectively. I will write down the difference in profit between bidding at $x_j(b)$ and b for bidder j in three parts corresponding to $\Pi_j^-(x_i(b)) - \Pi_j(x_j(b))$, $\Pi_j^+(x_i(b)) - \Pi_j^-(x_i(b))$, and $\Pi_j(b) - \Pi_j^+(x_i(b))$:

$$\begin{aligned}\Pi_j(b) - \Pi_j(x_j(b)) &= (G_i(x_j(b)) \Pr[H_i|H_j] + \Pr[L_i|H_j]) \int_{x_j(b)}^{x_i(b)} \left(b_j^*(\Gamma_1) - t\right) \hat{r}(t)\, dt \\ &\quad + \Pr[H_i|H_j] (G_i(x_i(b)) - G_i(x_j(b))) \hat{R}(x_i(b)) (1 - x_i(b)) \\ &\quad + (G_i(x_i(b)) \Pr[H_i|H_j] + \Pr[L_i|H_j]) \int_{x_i(b)}^{b} \left(b_j^*(\Gamma_2) - t\right) \hat{r}(t)\, dt\end{aligned}$$

The third term $\Pi_j(b) - \Pi_j(x_i(b))$ is positive since $b = b_j^*(\Gamma_2)$ implies the following integral is positive:

$$\int_{x_i(b)}^{b} \left(b_j^*(\Gamma_2) - t\right) \hat{r}(t)\, dt = \int_{x_i(b)}^{b} (b - t) \hat{r}(t)\, dt > 0. \tag{7}$$

The fact that $b_j^*(\Gamma_1) = x_j(b)$ provides a lower bound to the integral in the first term:

$$\int_{x_j(b)}^{x_i(b)} \left(b_j^*(\Gamma_1) - t\right) \hat{r}(t)\, dt \geq -\left(\hat{R}(x_i(b)) - \hat{R}(x_j(b))\right) (x_i(b) - x_j(b)) \geq -\hat{R}(x_i(b)) (x_i(b) - x_j(b)). \tag{8}$$

The inequalities in equations (7) and (8) imply that

$$\begin{aligned}\Pi_j(b) - \Pi_j(x_j(b)) &> -(G_i(x_j(b)) \Pr[H_i|H_j] + \Pr[L_i|H_j]) \hat{R}(x_i(b)) (x_i(b) - x_j(b)) & (9)\\ &\quad + \Pr[H_i|H_j] (G_i(x_i(b)) - G_i(x_j(b))) \hat{R}(x_i(b)) (1 - x_i(b)) & (10)\end{aligned}$$

Substituting $x_j(b) = b_j^*(G_i(x_j(b))) = \frac{G_i(x_j(b))\Pr(H_i|H_j)+\Pr(L_i|H_j)v_j}{G_i(x_j(b))\Pr(H_i|H_j)+\Pr(L_i|H_j)}$ into the right-hand side of equation (10) and canceling and regrouping terms gives

$$\hat{R}(x_i(b))(G_i(x_i(b))\Pr(H_i|H_j) + \Pr(L_i|H_j))\left(\frac{G_i(x_i(b))\Pr(H_i|H_j) + \Pr(L_i|H_j)v_j}{(G_i(x_i(b))\Pr(H_i|H_j) + \Pr(L_i|H_j))} - x_i(b)\right).$$

Finally, since $G_i(x_i(b)) = G_i(b)$ and $b = b_j^*(G_i(b))$ we can substitute in $b = \frac{G_i(x_i(b))\Pr(H_i|H_j)+\Pr(L_i|H_j)v_j}{G_i(x_i(b))\Pr(H_i|H_j)+\Pr(L_i|H_j)}$ yielding

$$\hat{R}(x_i(b))(G_i(x_i(b))\Pr(H_i|H_j) + \Pr(L_i|H_j))(b - x_i(b)),$$

which is positive since $b > x_i(b)$. Thus $\Pi_j(b) - \Pi_j(x_j(b)) > 0$. □

Recall the definition $b_{min} = \max\{\underline{b}_1, \underline{b}_2\}$. In addition, define $\bar{b}_i = \inf\{x : G_i(x) = 1\}$ and $\bar{b}_j = \inf\{x : G_j(x) = 1\}$. Finally, define $b_{max} = \max\{\bar{b}_1, \bar{b}_2\}$. Notice that $b_{max} \geq b_{min}$.

Lemma 51. *At η the following must hold.*

1. *If $b_{max} > b_{min}$ then both bidders have the same supremum bid: $\bar{b}_i = \bar{b}_j = b_{max}$.*
2. *Both G_1 and G_2 are continuous for all $b > b_{min}$. Moreover, both G_1 and G_2 are strictly increasing over the interval (b_{min}, b_{max}).*
3. *Suppose that $\underline{b}_i > \underline{b}_j$ so that $b_{min} = \underline{b}_i > \underline{b} = \underline{b}_j$. Then j bids an atom at $\underline{b} = \underline{b}_j = v_j$ and i bids an atom at $b_{min} = \underline{b}_i = b_i^*(G_j(v_j))$ with weight Γ_i. Moreover the size of i's atom at b_{min} is 1 if $b_{max} = b_{min}$ and otherwise is:*

$$\Gamma_i = \frac{\Pr[L_i|H_j]}{\Pr[H_i|H_j]}\frac{\int_{v_j}^{b_{min}}(x - v_j)\hat{r}(x)\,dx}{\hat{R}(b_{min})(1 - b_{min})}$$

Proof. (1) Suppose not and $b_{max} = \bar{b}_i > \bar{b}_j$. Then j does not bid over $(\bar{b}_j, \bar{b}_i)$ but i bids with positive probability in $(\bar{b}_j, \bar{b}_i]$. By Lemma 43 and the definition of $\bar{b}_i$, this positive probability must be concentrated at a single atom at $\bar{b}_i$. By Lemma 47, $\bar{b}_i$ is i's infimum bid, that is $\bar{b}_i = \underline{b}_i$, thus $\bar{b}_i = \underline{b}_i \leq b_{min} \leq b_{max} = \bar{b}_i$, so $b_{min} = b_{max}$, a contradiction.

(2) By Lemma 50 G_i and G_j are continuous for all $b > b_{min}$. To show that they must also be strictly increasing over (b_{max}, b_{min}) we consider and rule out two types of flat spots. Throughout, we assume $b_{max} > b_{min}$ (the claim is trivially satisfied for $b_{max} = b_{min}$).

First, suppose that at least one bidder, say i, does not bid in an interval (b_{min}, b^+) so that $G_i(b_{min}) = G_i^-(b^+) = \Gamma$ for some $b^+ > b_{min}$. Note that $b_{max} > b_{min}$ and part (1) imply $\Gamma < 1$ and no atoms above b_{min} implies $G_i^-(b^+) = G_i(b^+)$. Moreover, let b^+ be the upper bound of the flat spot: $b^+ = sup\{b : G_i(b) = G_i(b_{min})\}$. By Lemma 43, j can place at most one bid over $(b_{min}, b^+]$. By definition, b_{min} must be the infimum bid of one or both bidders. As neither bidder bids in (b_{min}, b^+), this implies one (but not both by Lemma 43) bidders has an atom at b_{min}. By the definition of b^+ and the fact that j does not bid an atom at b^+, b^+ must be an optimal bid for i.

Suppose (i) i has the atom at b_{min}. Then i has optimal bids at b_{min} and b^+ but $G_j(b_{min}) = G_j(b^+)$, contradicting Lemma 46.

Suppose instead (ii) that j has the atom at b_{min}. By Lemma 46, b^+ is not an optimal bid for j because b_{min} is optimal but $G_i(b_{min}) = G_i(b^+)$. Because i does not bid an atom at b^+, $\Pi_j(b)$ is continuous at b^+ and j does not have an optimal bid in a neighborhood $(b^+ - \beta, b^+ + \beta)$ for $\beta > 0$ sufficiently small. However i must bid with positive probability in this interval by definition of b^+, by Lemma 43 it must be concentrated at an atom, and this contradicts no atoms above b_{min}.

Second, suppose that at least one bidder, say i, does not bid in an interval (b^-, b^+) such that $G_i(b^-) = G_i^-(b^+) = \Gamma$ where

$$b_{min} < b^- = inf\{b : G_i(b) = \Gamma\} < b^+ = sup\{b : G_i(b) = \Gamma\} < b_{max}.$$

Note that $b^- > b_{min}$ implies $\Gamma > 0$ and $b_{max} > b^+$ implies $\Gamma < 1$. Because there are no atoms above b_{min}, both agent's utility functions are continuous at b^- and b^+. Thus the definitions of b^- and b^+ (and $\Gamma \in (0, 1)$) therefore imply that b^- and b^+ are both optimal bids for i. By Lemma 43, j can place at most one bid over $(b^-, b^+]$, and because j cannot have an atom, this implies $G_j(b^-) = G_j(b^+)$. By Lemma 46, this contradicts optimality of b^- and b^+ for i.

(3) Lemma 47 and $\underline{b}_j < \underline{b}_i$ imply that j bids an atom at $\underline{b} = v_j$ but nowhere else below b_{min}. The final step in the proof is to show that i bids an atom at b_{min}. Then Corollary 44 implies $b_{min} = b_i^*(G_j(b_{min}))$. Finally $G_j(b_{min}) = G_j(v_j)$ because j does not bid in $(v_j, b_{min}]$ (Lemmas 40 and 47).

To show that i bids an atom at b_{min}, there are two cases. (1) $b_{max} = b_{min}$: This implies that j's atom at v_j has mass 1 and that i bids b_{min} with probability 1. (2) $b_{max} > b_{min}$: Then by part (1) of this Lemma, for any $\delta > 0$ bidder j has optimal bid within the interval $(b_{min}, b_{min} + \delta)$. This means that bidder i must have an atom at $b_{min} = \underline{b}_i$. Suppose not and $G_i(b_{min}) = G_i(0)$. Then b_{min} will be an optimal bid for j by continuity but $\underline{b}$ is also an optimal bid for j. This contradicts Lemma 46 given $G_i(b_{min}) = G_i(0)$.

To compute Γ_i we observe that the utility of j is the same across all bids in the support, in particular at his atom at $\underline{b}_j = v_j$ and at any optimal bid $b_j > b_{min}$ that is arbitrarily close to b_{min} (such bid exists for any $\delta > 0$ in the interval $(b_{min}, b_{min} + \delta)$ since $b_{max} > b_{min}$). Thus the change in utility from increasing the bid from v_j to such b_j is zero. The next equation presents this utility change in the limit when b_j tends to b_{min} from above.

$$\hat{R}(b_{min}) \Pr[H_i|H_j]\Gamma_i(1 - b_{min}) - \Pr[L_i|H_j] \int_{v_j}^{b_{min}} (x - v_j)\, \hat{r}\,(x)\, dx = 0$$

Or equivalently,

$$\Gamma_i = \frac{\Pr\left[L_i|H_j\right]}{\Pr\left[H_i|H_j\right]} \frac{\int_{v_j}^{b_{min}} (x - v_j)\, \hat{r}\,(x)\, dx}{\hat{R}\,(b_{min})\,(1 - b_{min})}$$

□

Recall that $\underline{b}_i = \inf\{b : G_i(b) > 0\}$ and $\bar{b}_i = \inf\{x : G_i(x) = 1\}$ for agent $i \in \{1, 2\}$. Note that when agent never submit dominated bids by definition it holds that $1 \geq b_{max} = \max\{\bar{b}_i, \bar{b}_j\} \geq b_{min} = \max\{\underline{b}_1, \underline{b}_2\} \geq \underline{b} = \min\{\underline{b}_1, \underline{b}_2\} \geq 0$.

We are now ready to restate Lemma 20 and prove it.

Lemma 52. *At η the following must hold.*

1. *For some $j \in \{1, 2\}$ it holds that $\underline{b} = \underline{b}_j = v_j$ and $b_{min} = \underline{b}_i \geq v_i$ for $i \neq j$.*
2. *Both G_1 and G_2 are continuous and strictly increasing on (b_{min}, b_{max}). It holds that $G_1(b_{max}) = G_2(b_{max}) = 1$. Moreover, if $b_{max} > b_{min}$ then $b_{max} = \bar{b}_1 = \bar{b}_2$.*
3. *For every bidder $i \in \{1, 2\}$ it holds that $G_i(b) = 0$ for every $b \in [0, \underline{b})$, and $G_i(b) = G_i(\underline{b})$ for every $b \in [\underline{b}, b_{min})$.*

4. *If $b_{min} = \underline{b}$ then $\underline{b} = \max\{v_1, v_2\}$. Additionally, if $v_1 = v_2$ then $b_{min} = \underline{b} = v_1 = v_2$ and no bidder has any atom anywhere. If $v_i > v_j$ then $b_{min} = \underline{b} = v_i$ and i has an atom at $\underline{b}$, while j has no atoms.*
5. *If $b_{min} > \underline{b}$ then for one agent, say j, it holds that $\underline{b} = \underline{b}_j = v_j$. Bidder j has an atom at v_j and bidder $i \neq j$ has an atom at*

$$b_{min} = b_i^*(G_j(v_j)) = \frac{\Pr[H_j|H_i]G_j(v_j) + v_i \Pr[L_j|H_i]}{\Pr[H_j|H_i]G_j(v_j) + \Pr[L_j|H_i]} > \max\{v_i, v_j\} \tag{11}$$

and b_{min} satisfies $b_{min} \leq v(H_i)$, and $b_{min} = v(H_i)$ if and only if $G_j(v_j) = 1$.
It also holds that either

- *$b_{max} = b_{min}$, in this case $G_i(b_{min}) = 1$, $G_j(v_j) = 1$ (j always bids v_j, i always bids b_{min}). Or*
- *$b_{max} > b_{min}$, $G_i(b_{min}) > 0$ and*

$$G_i(b_{min}) = \frac{\Pr[L_i|H_j]}{\Pr[H_i|H_j]} \frac{\int_{v_j}^{b_{min}} (x - v_j)\hat{r}(x)\,dx}{\hat{R}(b_{min})(1 - b_{min})} \tag{12}$$

Proof. (1) Follows from Lemma 47. (2) Follows from the definition of b_{max} and Lemma 51 parts 1 and 2. (3) $G_i(b) = 0$ for $b < \underline{b}$ follows from the definition of $\underline{b}$. $G_i(b) = G_i(\underline{b})$ for $b \in [\underline{b}, b_{min})$ follows from Lemma 47 part 2. (4) Follows from Lemma 47 part 1. (5) Follows almost entirely from Lemma 51 part 3. The fact that $\max\{v_i, v_j\} < b_{min} \leq v(H_i)$ and $b_{min} = v(H_i)$ if and only if $G_j(v_j) = 1$ follows from the definition of b_{min}, inspection of equation (11), and the fact that $v(H_i) = \Pr[H_j|H_i] + v_i \Pr[L_j|H_i]$. □

Lemma 53. *At η the following must hold. If $\epsilon > 0$ is small enough then $b_{max} > b_{min}$.*

Proof. Assume that $b_{max} = b_{min}$. Clearly it cannot be the case that $b_{min} = \underline{b}$ as it means that both agents are bidding an atom (of size 1) at $\underline{b}$. If $b_{min} < 1$, this contradicts Lemma 40. If $b_{min} \geq 1$, bidder $i \in \{1, 2\}$ could earn strictly more by deviating to bid v_i. Reducing the bid to v_i means that bidder i loses every time bidder $j \neq i$ has a high signal. In these cases the value is 1, but the payment would have been 1, so bidder i is indifferent to losing rather than tying. In addition, reducing the bid to v_i means that bidder i now loses every time that bidder j has a low signal and the random bidder bids between v_i and i. Thus the bid reduction avoids overpayment with positive probability. This contradicts optimality of bidder i bidding 1. We conclude that $b_{min} > \underline{b}$.

Given $b_{max} = b_{min} > \underline{b}$, Lemma 20 implies that one agent, say j, is bidding an atom of size 1 at v_j, while the other agent i is bidding an atom of size 1 at $b_{min} = b_i^*(1)$. We note that Equation (11) shows that for $v_i < 1$ there exists $\zeta < 1$ which is independent of ϵ such that $b_{min} < \zeta$. When ϵ is small enough agent j can deviate and get strictly higher utility by bidding $b^+ \in (b_{min}, 1)$. This deviation has two effects. First it means that j has additional wins when i has a low signal and the random bidder bids between v_j and b^+ causing j to pay more than the value v_j. This costs bidder j

$$\epsilon \Pr[L_i|H_j] \int_{v_j}^{b^+} (x - v_j)\, r(x)\, dx < \epsilon$$

which is proportional to ε. In addition, the deviation means that j has additional wins when i has a high signal and the random bidder bids below b^+. All of these incremental wins are valued at 1 but cost no more than b^+ so increase j's payoff. Considering just those incremental wins for which the random bidder bids below b_{min}, this benefit is bounded below by $\Pr[H_i|H_j](1-b_{min}) > \Pr[H_i|H_j](1-\zeta)$. Thus $\epsilon < \Pr[H_i|H_j](1-\zeta)$ is a sufficient condition for the deviation to be strictly profitable. This contradiction shows $b_{max} > b_{min}$. □

D.2 Proofs of Lemma 21 and of Theorem 18

We next restate Theorem 18 and prove it.

Theorem 54. *Consider any non-degenerated monotonic domain with 2 bidders, each with a binary signal. Assume that* $0 < Pr[H_1, L_2](1 - v(H_1, L_2)) \leq Pr[L_1, H_2](1 - v(L_1, H_2)) < 1$.

The unique TRE of the SPA game is the profile of strategies μ *in which:*

- *Every bidder* i *bids* $v(L_1, L_2) = 0$ *when getting signal* L_i.
- *Bidder* 1 *with signal* H_1 *always bids* $v(H_1, H_2) = 1$.
- *Bidder* 2 *with signal* H_2
  - *bids* $v(H_1, H_2) = 1$ *with probability* $\frac{Pr[H_1,L_2]}{Pr[L_1,H_2]} \cdot \frac{1-v(H_1,L_2)}{1-v(L_1,H_2)}$*, and*
  - *bids* $v(L_1, H_2)$ *with the remaining probability.*

Recall that to simplify the notation we denote $v_1 = v(H_1, L_2)$ and $v_2 = v(L_1, H_2)$. As the domain is non-degenerated, $Pr[H_1, H_2] > 0$ and for any bidder $i \in \{1, 2\}$ it holds that $1 > Pr[H_i] > 0$. The assumption that $0 < Pr[H_1, L_2](1-v_1) \leq Pr[L_1, H_2](1-v_2) < 1$ implies $\max\{v_1, v_2\} < 1$ and that $\min\{Pr[H_1, L_2], Pr[L_1, H_2]\} > 0$ and combining with the above implies that $\min\{Pr[L_2|H_1], Pr[L_2|H_1]\} > 0$. Additionally, as the domain is non-degenerated, $\min\{Pr[H_1|H_2], Pr[H_2|H_1]\} > 0$.

Consider the game with the random bidder that is bidding according to a standard distribution (its support is $[0, 1]$). The random bidder arrives to the auction with small probability $\epsilon > 0$.

Assume that agent i with signal H_i is bidding according to distribution G_i, let g_i denote the density of G_i whenever G_i is differentiable (note that since G_i is non-decreasing it is differentiable almost everywhere, see, for example, Theorem 31.2 in (Billingsley 1995)). We note that this is an abuse of notation as G_i and g_i both depend on R and ϵ.

To prove the theorem we show that for any standard distribution R and small enough ϵ a mixed NE in each of the games $\lambda(\epsilon, R)$ exists (Lemma 69). We then show that the limit of any sequence of NE strategies in the games $\lambda(\epsilon, R)$ must converges to μ as ϵ goes to zero. Combined with the existence of a mixed NE in each of the games $\lambda(\epsilon, R)$ this show that μ is the limit of the some sequence of NE strategies in the games $\lambda(\epsilon, R)$, thus a TRE. As the limit of any sequence of NE strategies in the games $\lambda(\epsilon, R)$ must converges to μ as ϵ goes to zero, μ is the *unique* TRE.

Fix a standard distribution R and $\epsilon > 0$ and consider the game $\lambda(\epsilon, R)$. Let $\eta = (G_1, G_1^L, G_2, G_2^L)$ be a NE of $\lambda(\epsilon, R)$. For agent i let $\underline{b}_i = \inf\{b : G_i(b) > 0\}$. Define $\underline{b} = \min\{\underline{b}_1, \underline{b}_2\}$ and $b_{min} = \max\{\underline{b}_1, \underline{b}_2\}$.

Lemma 20 characterizes candidates for NE in $\lambda(\epsilon, R)$. We next take it as given and defer the proof of it to Section D.1.

The following two well known theorems (see for example (Billingsley 1995)) will be useful for proving our lemmas.

Theorem 55 (Theorem 31.2 in (Billingsley 1995)). *A non-decreasing function G is differentiable almost everywhere, the derivative g is non-negative, and $G(b) - G(a) \geq \int_a^b g(x)dx$ for all a and b.*

Theorem 56 (Theorem 31.3 in (Billingsley 1995)). *If g is non-negative and integrable, and if $G(b) = \int_{-\infty}^b g(x)dx$, then $\frac{\partial G(b)}{\partial b} = g(b)$ except on a set of Lebesgue measure 0.*

In addition, the following well known differential-equation result is also useful for proving our lemmas.

Theorem 57. *Assume that $q(x) = u'(x) + p(x) \cdot u(x)$ holds for every $x \in (b_{min}, b)$ but a set of measure zero, and $p(x)$ and $q(x)$ are continuous on the interval. Define $z(x) = e^{\int_{b_{min}}^x p(y)dy}$. Then every function $u(x)$ that satisfies the assumption is of the form*

$$u(b) - \frac{u(b_{min})}{z(b)} = \frac{1}{z(b)} \int_{b_{min}}^b z(x)q(x)dx + C \tag{13}$$

for some C.

D.2.1 Characterizations of the CDFs of G_1 and G_2

Lemma 58. *At η the following must hold. For every bid $b \geq b_{min}$ in the support of agent 2's distribution G_2, it must holds that*

$$Pr[L_1|H_2] \cdot \epsilon \int_{b_{min}}^b (x - v_2) \cdot r(x)dx = Pr[H_1|H_2] \left(s_1^{(min)}(b) + s_1^{(+)}(b) \right) \tag{14}$$

where

$$s_1^{(min)}(b) = \epsilon \cdot G_1(b_{min}) \int_{b_{min}}^b (1 - y) \cdot r(y)dy \tag{15}$$

and

$$s_1^{(+)}(b) = \int_{b_{min}}^b g_1(x) \left(\hat{R}(b)(1 - x) - \epsilon \cdot \int_x^b (y - x)r(y)dy \right) dx \tag{16}$$

Proof. By Lemma 20 for any $\delta > 0$ agent 2 has an optimal bid in $[b_{min}, b_{min}+\delta)$. Agent 2 must be indifferent between all his bids, in particular, between bidding b and bidding arbitrarily close to b_{min}. The left hand side is the decrease in the expected utility of agent 2 when the agent 1 receives signal L_1 (happens with probability $Pr[L_1|H_2]$). As agent 1 with signal L_1 bids 0 (Lemma 39), agent 2 with signal H_2 bidding a positive value always beats agent 1. Any time agent 2's wins he gets a value of v_2. A bid of b wins while a bid arbitrary close to b_{min} does not, only when the random bidder arrives (happens with probability ϵ). In this case the extra utility gain is $\int_{b_{min}}^b (x - v_2) \cdot r(x)dx$.

The right side handles the net gain when agent 1 receives signal H_1 (happens with probability $Pr[H_1|H_2]$). Agent 1 is bidding at most b_{min}, which happen with probability $G_1(b_{min})$. By bidding b and not arbitrarily close to b_{min} the presence of the random bidder creates an additional utility of $s_1^{(min)}(b) = \epsilon \cdot G_1(b_{min}) \int_{b_{min}}^b (1 - y) \cdot r(y)dy$ to agent 2.

Next consider the case that agent 1 is bidding more than b_{min}. With probability $1 - \epsilon$ the random bidder is bidding 0. By Lemma 20, G_1 is continuous for every $b \in (b_{min}, 1)$ (has no atom at bid in $(b_{min}, 1)$). As G_1 is continuous for every $b \in (b_{min}, 1)$, the expected utility is given by $\int_{b_{min}}^b g_1(x) (1 - x) dx$, as the value from winning is 1, the payment if set by agent 1 is

x. (note that by Theorem 56 G_1 if differentiable at all points but a set of measure 0, and there are no atoms at these points, thus this set cannot change the integral). Thus conditional on H_1, the contribution of this case to the utility of agent 2 is

$$(1-\epsilon)\int_{b_{min}}^{b} g_1(x)\,(1-x)\,dx \tag{17}$$

Additionally, with probability ϵ the random bidder is bidding according to R. In this case the utility difference is

$$\epsilon\left(\int_{b_{min}}^{b} g_1(x)\left(\int_{b_{min}}^{x}(1-x)r(y)dy+\int_{x}^{b}(1-y)r(y)dy\right)dx\right)= \tag{18}$$

$$\epsilon\left(\int_{b_{min}}^{b} g_1(x)\left(R(x)(1-x)+\int_{x}^{b}(1-x+x-y)r(y)dy\right)dx\right)= \tag{19}$$

$$\epsilon\left(\int_{b_{min}}^{b} g_1(x)\left(R(b)(1-x)-\int_{x}^{b}(y-x)r(y)dy\right)dx\right) \tag{20}$$

Now Equation 14 is derived by noting that $\hat{R}(x)=1-\epsilon+\epsilon\cdot R(x)$.

□

Lemma 59. *At η the following must hold. For every bid $b\geq b_{min}$ in the support of agent 2's distribution G_2, if $\frac{\partial s_1^{(+)}(b)}{\partial b}$ exists at b then it holds that*

$$\frac{Pr[L_1|H_2]}{Pr[H_1|H_2]}\cdot\frac{b-v_2}{1-b}\cdot\frac{\hat{r}(b)}{\hat{R}(b)}=g_1(b)+\frac{\hat{r}(b)}{\hat{R}(b)}\cdot G_1(b) \tag{21}$$

Proof. Under the condition of the lemma, by Lemma 58 we know that Equation (14) holds at b. We show that if $\frac{\partial s_1^{(+)}(b)}{\partial b}$ exists we can take the derivative of Equation (14) with respect to b and that this yield the desired equality.

First we look at the LHS of Equation (14). As R is standard r is continuous and the derivative with respect to b exists. It holds that the derivative of the LHS of Equation (14) is

$$\frac{\partial(Pr[L_1|H_2]\cdot\epsilon\int_{b_{min}}^{b}(x-v_2)\cdot r(x)dx)}{\partial b}=Pr[L_1|H_2]\cdot\epsilon\cdot(b-v_2)\cdot r(b) \tag{22}$$

The derivative of the RHS of Equation (14) exists as $\frac{\partial s_1^{(+)}(b)}{\partial b}$ exists by assumption and $\frac{\partial s_1^{(min)}(b)}{\partial b}$ exists as r is continuous. The derivative is

$$Pr[H_1|H_2]\cdot\frac{\partial\left(s_1^{(min)}(b)+s_1^{(+)}(b)\right)}{\partial b}=Pr[H_1|H_2]\cdot\left(\frac{\partial s_1^{(min)}(b)}{\partial b}+\frac{\partial s_1^{(+)}(b)}{\partial b}\right) \tag{23}$$

Now

$$\frac{\partial s_1^{(min)}(b)}{\partial b}=\epsilon\cdot G_1(b_{min})\cdot(1-b)\cdot r(b) \tag{24}$$

and

$$\frac{\partial s_1^{(+)}(b)}{\partial b}=\frac{\partial\left(\int_{b_{min}}^{b}g_1(x)\left(\hat{R}(b)(1-x)-\epsilon\cdot\int_{x}^{b}(y-x)r(y)dy\right)dx\right)}{\partial b}= \tag{25}$$

$$\frac{\partial\left(\hat{R}(b)\int_{b_{min}}^{b} g_1(x)\,(1-x)\,dx\right)}{\partial b} - \epsilon\frac{\partial\left(\int_{b_{min}}^{b} g_1(x)\left(\int_x^b (y-x)r(y)dy\right)dx\right)}{\partial b} \tag{26}$$

We handle each of the two terms:

$$\frac{\partial\left(\hat{R}(b)\int_{b_{min}}^{b} g_1(x)\,(1-x)\,dx\right)}{\partial b} = \epsilon\cdot r(b)\left(\int_{b_{min}}^{b} g_1(x)\,(1-x)\,dx\right) + \hat{R}(b)\cdot g_1(b)\,(1-b) \tag{27}$$

Now define $h(b,x) = \int_x^b (y-x)r(y)dy$ and note that $h(b,b) = 0$ and $\frac{\partial h(b,x)}{\partial b} = (b-x)r(b)$

$$\frac{\partial \int_{b_{min}}^{b} g_1(x)h(b,x)dx}{\partial b} = \int_{b_{min}}^{b} g_1(x)\frac{\partial h(b,x)}{\partial b}dx = r(b)\cdot\int_{b_{min}}^{b} g_1(x)(b-x)dx \tag{28}$$

We conclude that

$$\frac{\partial s_1^{(+)}(b)}{\partial b} = \epsilon\cdot r(b)\int_{b_{min}}^{b} g_1(x)\,(1-x)\,dx + \hat{R}(b)\cdot g_1(b)\,(1-b) - \epsilon\cdot r(b)\cdot\int_{b_{min}}^{b} g_1(x)(b-x)dx = \tag{29}$$

$$\epsilon\cdot r(b)\int_{b_{min}}^{b} g_1(x)\,(1-b)\,dx + \hat{R}(b)\cdot g_1(b)\,(1-b) = \tag{30}$$

$$\epsilon\cdot r(b)(G_1(b) - G_1(b_{min}))\,(1-b) + \hat{R}(b)\cdot g_1(b)\,(1-b) \tag{31}$$

Summarizing:

$$\frac{\partial s_1^{(min)}(b)}{\partial b} + \frac{\partial s_1^{(+)}(b)}{\partial b} = \epsilon\cdot r(b)G_1(b)\,(1-b) + \hat{R}(b)\cdot g_1(b)\,(1-b) \tag{32}$$

Combining all the above with the observation that $\hat{r}(b) = \epsilon\cdot r(b)$ we conclude that

$$\frac{Pr[L_1|H_2]}{Pr[H_1|H_2]}\cdot(b-v_2)\cdot\hat{r}(b) = \hat{r}(b)\cdot G_1(b)\,(1-b) + \hat{R}(b)\cdot g_1(b)\,(1-b) \tag{33}$$

Equivalently, by reorganizing, this yields equation (21). □

Lemma 60. *At η the following must hold. For every bid $b \geq b_{min}$ in the support of agent 2's distribution G_2, it must holds that*

$$G_1(b) = \frac{Pr[L_1|H_2]}{Pr[H_1|H_2]}\cdot\frac{\epsilon}{\hat{R}(b)}\cdot\int_{b_{min}}^{b}\frac{x-v_2}{1-x}r(x)dx + G_1(b_{min})\cdot\frac{\hat{R}(b_{min})}{\hat{R}(b)} \tag{34}$$

Proof. By Theorem 55 G_1 is differentiable almost everywhere. At any point x for which G_1 is differentiable ($g_1(x)$ exists) it holds that $\frac{\partial s_1^{(+)}(b)}{\partial b}$ exists. By Lemma 59 if $\frac{\partial s_1^{(+)}(b)}{\partial b}$ exists for an optimal $b \in (b_{min}, 1)$, then for b it holds that

$$\frac{Pr[L_1|H_2]}{Pr[H_1|H_2]}\cdot\frac{b-v_2}{1-b}\cdot\frac{\hat{r}(b)}{\hat{R}(b)} = g_1(b) + \frac{\hat{r}(b)}{\hat{R}(b)}\cdot G_1(b) \tag{35}$$

This is a First-Order Ordinary Differential Equation. We apply Theorem 57 with $u(b) = G_1(b)$, $u'(b) = g_1(b)$, $q(b) = \frac{Pr[L_1|H_2]}{Pr[H_1|H_2]}\cdot\frac{b-v_2}{1-b}\cdot\frac{\hat{r}(b)}{\hat{R}(b)}$ and $p(b) = \frac{\hat{r}(b)}{\hat{R}(b)}$.

We observe that $z(x) = \int_{b_{min}}^{x} p(y)dy = \int_{b_{min}}^{x} \frac{\hat{r}(y)}{\hat{R}(y)}dy = log(\hat{R}(x)) - log(b_{min})$, thus $z(x) = e^{\int_{b_{min}}^{x} p(y)dy} = \hat{R}(x)/\hat{R}(b_{min})$. Thus

$$G_1(b) - G_1(b_{min})\frac{\hat{R}(b_{min})}{\hat{R}(b)} = \frac{\hat{R}(b_{min})}{\hat{R}(b)}\left(\int_{b_{min}}^{b} \frac{\hat{R}(x)}{\hat{R}(b_{min})}\frac{Pr[L_1|H_2]}{Pr[H_1|H_2]} \cdot \frac{x - v_2}{1 - x} \cdot \frac{\hat{r}(x)}{\hat{R}(x)}dx + C\right) = \tag{36}$$

$$\frac{1}{\hat{R}(b)}\left(\int_{b_{min}}^{b} \frac{Pr[L_1|H_2]}{Pr[H_1|H_2]} \cdot \frac{x - v_2}{1 - x} \cdot \hat{r}(x)dx + C\right) \tag{37}$$

C is determined by evaluating the above at $b = b_{min}$. It must hold that $C = 0$. We conclude that

$$G_1(b) = \frac{Pr[L_1|H_2]}{Pr[H_1|H_2]} \cdot \frac{\epsilon}{\hat{R}(b)} \int_{b_{min}}^{b} \frac{x - v_2}{1 - x} \cdot r(x)dx + G_1(b_{min}) \cdot \frac{\hat{R}(b_{min})}{\hat{R}(b)} \tag{38}$$

□

By replacing each player by the other and repeating the proof above similarly we get:

Lemma 61. *At η the following must hold. For every bid $b \geq b_{min}$ in the support of agent 1's distribution G_1, it must holds that*

$$G_2(b) = \frac{Pr[L_2|H_1]}{Pr[H_2|H_1]} \cdot \frac{\epsilon}{\hat{R}(b)} \int_{b_{min}}^{b} \frac{x - v_1}{1 - x} \cdot r(x)dx + G_2(b_{min}) \cdot \frac{\hat{R}(b_{min})}{\hat{R}(b)} \tag{39}$$

D.2.2 Proofs of Lemma 21

We first show that b_{max} tends to 1 as ϵ goes to 0.

Lemma 62. *Fix a small $\delta > 0$. At η the following must hold. If $\epsilon > 0$ is small enough then it holds that $1 > b_{max} > 1 - \delta$ (b_{max} tends to 1 as ϵ goes to 0).*

Proof. By Lemma 60 and Lemma 61, for each bidder $i \in \{1, 2\}$ and $j \neq i$, b_{max} must satisfy:

$$1 = \frac{Pr[L_i|H_j]}{Pr[H_i|H_j]} \cdot \frac{\epsilon}{\hat{R}(b_{max})} \int_{b_{min}}^{b_{max}} \frac{x - v_i}{1 - x} \cdot r(x)dx + G_i(b_{min}) \cdot \frac{\hat{R}(b_{min})}{\hat{R}(b_{max})} \tag{40}$$

As ϵ approaches zero, Lemma 53, Lemma 20, and Equation (12) imply that for some bidder i either $G_i(b_{min}) = 0$ or $G_i(b_{min})$ approaches zero. For the first term to approach 1 as ϵ approaches zero requires the integral to approach infinity. As r is continuous on a compact set its infimum is obtained. Since r is positive for every x, $\exists \underline{r} > 0$ such that $r(x) \geq \underline{r}$ for every x. Hence it is clear that for the integral to approach infinity b_{max} must approach 1. Finally, for any fixed $\epsilon > 0$ it must hold that $1 > b_{max}$ as the integral to approach infinity as b_{max} tends to 1.

□

The following notations will be useful. Let $\alpha_1 = \frac{Pr[L_1|H_2]}{Pr[H_1|H_2]}$ and $\alpha_2 = \frac{Pr[L_2|H_1]}{Pr[H_2|H_1]}$.

We assume that $0 < Pr[H_1, L_2](1 - v_1) \leq Pr[L_1, H_2](1 - v_2)$.[11] Additionally, assume that if $Pr[H_1, L_2](1 - v_1) = Pr[L_1, H_2](1 - v_2)$ then $v_1 \geq v_2$. The next lemma (Lemma 21) presents additional properties that candidates for NE must satisfy when ϵ is small enough.

[11]If $\min\{Pr[H_1, L_2](1 - v_1), Pr[L_1, H_2](1 - v_2)\} > 0$ this is without loss of generality, by renaming the bidders if necessary.

Lemma 63. *If ϵ is small enough at η the following must hold. There must exist b_{min} and b_{max} such that $1 > b_{max} > b_{min} \geq 0$ and:*

- *The two bidders are symmetric ($Pr[H_1, L_2] = Pr[L_1, H_2]$ and $v_1 = v_2$) if and only if $b_{min} = \underline{b} = v_1 = v_2$ and $G_1(b_{min}) = G_2(b_{min}) = 0$ (no atoms).*
- *If $Pr[H_1, L_2](1-v_1) = Pr[L_1, H_2](1-v_2)$ but the bidders are not symmetric, and it holds that $v_1 > v_2$ and $Pr[H_1, L_2] < Pr[L_1, H_2]$, then bidder 1 has an atom at $b_{min} = \underline{b}_1$ of size $G_1(b_{min}) > 0$, and bidder 2 has an atom at $v_2 = \underline{b}_2 = \underline{b} < b_{min}$ of size $G_2(v_2) > 0$. It holds that*

$$b_{min} = b_1^*(G_2(v_2)) = \frac{\Pr[H_2|H_1]G_2(v_2) + v_1 \Pr[L_2|H_1]}{\Pr[H_2|H_1]G_2(v_2) + \Pr[L_2|H_1]} > \max\{v_1, v_2\} \tag{41}$$

$$G_1(b_{min}) = \frac{\Pr[L_1|H_2]}{\Pr[H_1|H_2]} \frac{\int_{v_2}^{b_{min}} (x - v_2)\, \hat{r}\,(x)\, dx}{\hat{R}\,(b_{min})\,(1 - b_{min})} \tag{42}$$

$$G_2(v_2) = \frac{\hat{R}(b_{max})}{\hat{R}(b_{min})} - \left(\frac{\hat{R}(b_{max})}{\hat{R}(b_{min})} - G_1(b_{min}) \right) \cdot \frac{Pr[H_1, L_2]}{Pr[L_1, H_2]} \cdot \frac{\int_{b_{min}}^{b_{max}} \frac{x-v_1}{1-x} r(x)dx}{\int_{b_{min}}^{b_{max}} \frac{x-v_2}{1-x} r(x)dx} \tag{43}$$

- *Assume $Pr[H_1, L_2](1 - v_1) < Pr[L_1, H_2](1 - v_2)$. Then either*
  - *$b_{min} = \underline{b}$, bidder 1 has no atom ($G_1(b_{min}) = 0$) and bidder 2 has an atom at $\underline{b} = \underline{b}_2 = v_2 \geq v_1$ of size $G_2(v_2) > 0$ specified by Equation (43), or*
  - *$b_{min} > \underline{b}$, bidder 1 has an atom at $b_{min} = \underline{b}_1$ specified by Equation (41), its size $G_1(b_{min}) > 0$ is specified by Equation (42), and bidder 2 has an atom at $v_2 = \underline{b}_2 = \underline{b} < b_{min}$ of size $G_2(v_2) > 0$ specified by Equation (43).*

Moreover, it always hold that

$$G_1(b) = \begin{cases} 0 & \text{if } 0 \leq b < b_{min}; \\ \frac{Pr[L_1|H_2]}{Pr[H_1|H_2]} \cdot \frac{\epsilon}{\hat{R}(b)} \cdot \int_{b_{min}}^{b} \frac{x-v_2}{1-x} r(x)dx + G_1(b_{min}) \cdot \frac{\hat{R}(b_{min})}{\hat{R}(b)} & \text{if } b_{min} \leq b \leq b_{max}; \\ 1 & \text{if } b_{max} < b \leq 1. \end{cases} \tag{44}$$

and

$$G_2(b) = \begin{cases} 0 & \text{if } 0 \leq b < v_2; \\ G_2(v_2) & \text{if } v_2 \leq b < b_{min}; \\ \frac{Pr[L_2|H_1]}{Pr[H_2|H_1]} \cdot \frac{\epsilon}{\hat{R}(b)} \int_{b_{min}}^{b} \frac{x-v_1}{1-x} \cdot r(x)dx + G_2(v_2) \cdot \frac{\hat{R}(b_{min})}{\hat{R}(b)} & \text{if } b_{min} \leq b \leq b_{max}; \\ 1 & \text{if } b_{max} < b \leq 1. \end{cases} \tag{45}$$

Proof. By Lemma 53 for small enough ϵ it holds that $b_{max} > b_{min}$, so we assume that in the rest of the proof.

Observe that

$$\frac{\alpha_2}{\alpha_1} = \frac{Pr[L_2|H_1]}{Pr[L_1|H_2]} \cdot \frac{Pr[H_1|H_2]}{Pr[H_2|H_1]} = \frac{Pr[L_2|H_1]}{Pr[L_1|H_2]} \cdot \frac{Pr[H_1, H_2]}{Pr[H_2]} \cdot \frac{Pr[H_1]}{Pr[H_1, H_2]} = \frac{Pr[H_1, L_2]}{Pr[L_1, H_2]} \tag{46}$$

Let $T(b, b_{min}) = \frac{\int_{b_{min}}^{b} \frac{1}{1-x} \cdot r(x)dx}{\int_{b_{min}}^{b} \frac{x}{1-x} \cdot r(x)dx}$, and let $\beta(b) = \frac{\hat{R}(b_{min})}{\hat{R}(b)}$.

Claim 1. *Assume $b_{max} > b_{min}$. It holds that*

$$\frac{1 - G_2(b_{min}) \cdot \beta(b_{max})}{1 - G_1(b_{min}) \cdot \beta(b_{max})} = \frac{\alpha_2}{\alpha_1} \cdot \frac{1 - v_1 T(b_{max}, b_{min})}{1 - v_2 T(b_{max}, b_{min})} \tag{47}$$

Proof. Recall that by Lemma 20 for b_{max} it holds that $G_1(b_{max}) = G_2(b_{max}) = 1$.

By Lemma 61 for every bid $b \geq b_{min}$ in the support of agent 1's distribution G_1, that is, for every $b \in [b_{min}, b_{max})$, Equation (39) hold. As the equation is continuous at b_{max} we conclude that

$$1 - G_2(b_{min}) \cdot \beta(b_{max}) = \alpha_2 \cdot \frac{\epsilon}{\hat{R}(b_{max})} \int_{b_{min}}^{b_{max}} \frac{x - v_1}{1 - x} \cdot r(x) dx \tag{48}$$

Additionally, by Lemma 60 for every bid $b \geq b_{min}$ in the support of agent 2's distribution G_2, that is, for every $b \in [b_{min}, b_{max})$, Equation (34) hold. As the equation is continuous at b_{max} we conclude that

$$1 - G_1(b_{min}) \cdot \beta(b_{max}) = \alpha_1 \cdot \frac{\epsilon}{\hat{R}(b_{max})} \int_{b_{min}}^{b_{max}} \frac{x - v_2}{1 - x} \cdot r(x) dx \tag{49}$$

The claim follows from dividing the two equations (since for $b_{max} > b_{min}$ both sides of the two equations are not 0, thus such a division is well defined). □

Claim 2. *Assume $b_{max} > b_{min}$. There are no atoms ($G_1(b_{min}) = G_2(b_{min}) = 0$) if and only if both bidders are symmetric: $v_1 = v_2$ and $Pr[H_1, L_2] = Pr[L_1, H_2]$.*

Proof. By Lemma 20 if $G_1(b_{min}) = G_2(b_{min}) = 0$ then $\underline{b} = b_{min} = v_1 = v_2$. In such a case Equation (47) reduces to $\alpha_2 = \alpha_1$. Now, recall that $\frac{\alpha_2}{\alpha_1} = \frac{Pr[H_1,L_2]}{Pr[L_1,H_2]}$, thus if there are no atoms in both G_1 and G_2 then $\underline{b} = b_{min} = v_1 = v_2$ and $Pr[H_1, L_2] = Pr[L_1, H_2]$, that is, the two agents are completely symmetric.

Now, assume that both bidders are symmetric, that is, $v = v_1 = v_2$ and $Pr[H_1, L_2] = Pr[L_1, H_2]$, we want to show that no bidder has an atom. We next show that it cannot be the case that $b_{min} > \underline{b}$. This is sufficient as, by Lemma 20, $b_{min} = \underline{b}$ and $v_1 = v_2$ imply that no bidder has an atom, that is $G_2(b_{min}) = G_1(b_{min}) = 0$.

We next show that symmetry and $b_{min} > \underline{b}$ implies a contradiction. For symmetric bidders Equation (47) implies that $G_1(b_{min}) = G_2(b_{min})$. Using Lemma 20 we observe the following. One bidder, w.l.o.g. bidder 2, bids an atom at $\underline{b} = v_1 = v_2 = v$ and the other bidder (bidder 1) bids an atom at $b_{min} > \underline{b} = v$. Denote $\Gamma = G_1(b_{min}) = G_2(\underline{b})$. By Equation (41),

$$b_{min} = b_1^*(\Gamma) = \frac{\Pr[H_2|H_1]\,\Gamma + v_1 \Pr[L_2|H_1]}{\Pr[H_2|H_1]\,\Gamma + \Pr[L_2|H_1]},$$

or equivalently,

$$\Gamma = \frac{\Pr[L_2|H_1]}{\Pr[H_2|H_1]} \cdot \frac{b_{min} - v_1}{1 - b_{min}}.$$

By Equation (42),

$$\Gamma = \frac{\Pr[L_1|H_2]}{\Pr[H_1|H_2]} \cdot \frac{\int_{v_2}^{b_{min}} (x - v_2)\, \hat{r}(x) dx}{\hat{R}(b_{min})\,(1 - b_{min})}.$$

Thus,

$$\frac{\Pr[L_1|H_2]}{\Pr[H_1|H_2]} \cdot \frac{\int_{v_2}^{b_{min}} (x - v_2)\, \hat{r}(x) dx}{\hat{R}(b_{min})\,(1 - b_{min})} = \frac{\Pr[L_2|H_1]}{\Pr[H_2|H_1]} \cdot \frac{b_{min} - v_1}{1 - b_{min}},$$

or due to symmetry in conditional probabilities ($\alpha_1 = \alpha_2$) and values ($v_1 = v_2 = v$),

$$\int_v^{b_{min}} (x - v)\,\hat{r}(x)dx = \hat{R}\,(b_{min})\,(b_{min} - v)\,.$$

Integration by parts implies that

$$\int_v^{b_{min}} (x - v)\,\hat{r}(x)dx \quad = \quad (b_{min} - v)\,\hat{R}\,(b_{min}) - \int_v^{b_{min}} \hat{R}(x)dx,$$

and this can only equal $\hat{R}\,(b_{min})\,(b_{min} - v)$ when $b_{min} = v$, a contradiction. □

We next consider the case that $Pr[H_1, L_2](1 - v_1) = Pr[L_1, H_2](1 - v_2)$ but the bidders are not symmetric ($v_1 > v_2$ and $Pr[H_1, L_2] < Pr[L_1, H_2]$).

Claim 3. *Assume $b_{max} > b_{min}$ and that ϵ is small enough. Assume that $Pr[H_1, L_2](1 - v_1) = Pr[L_1, H_2](1 - v_2)$ but the bidders are not symmetric, and it holds that $v_1 > v_2$ and $Pr[H_1, L_2] < Pr[L_1, H_2]$. Then bidder 1 has an atom at $b_{min} = \underline{b}_1 > v_1$ and bidder 2 has an atom at $v_2 = \underline{b}_2 = \underline{b} < b_{min}$.*

Proof. By Claim 2 as bidders are not symmetric it cannot be the case that both bidders have no atom.

We next show that it cannot be the case that only one bidder has an atom. By Lemma 20 if only one bidder has an atom and $v_1 > v_2$ it must be the case that $\underline{b} = b_{min} = v_1 > v_2$ and bidder 1 has the atom at v_1. But in this case, as $G_2(b_{min}) = 0$, the LHS of Equation (47) equals to $\frac{1}{1 - G_1(b_{min}) \cdot \beta(b_{max})} > 1$ (as $0 < \beta(b_{max}) \leq 1$ and $G_1(b_{min}) > 0$), while the RHS of Equation (47) is at most 1 since by Lemma 64 it is monotonically increasing to its limit 1, a contradiction.

We conclude that both bidders have an atom, each at his infimum bid. We next figure out which bidder has an atom at $\underline{b}$ and which has an atom at b_{min}. We first show that it must be the case that both $G_1(b_{min})$ and $G_2(b_{min})$ tend to 0 as ϵ goes to 0. By Equation (12) for one bidder i it holds that $G_i(b_{min})$ must tend to 0 as ϵ goes to 0 (as b_{min} does not tend to 1 the denominator does not tend to 0, while the numerator tends to 0). Now, as the RHS of Equation (47) tends to 1 as ϵ goes to 0, $G_1(b_{min}) - G_2(b_{min})$ must tend to 0. Now, as both $G_1(b_{min})$ and $G_2(b_{min})$ tend to 0 as ϵ goes to 0, by Equation (11) the bid of bidder i that is bidding at $\underline{b}_i = b_{min}$ must tend to v_i, that is $b_{min} - v_i$ tends to 0. Now recall that in that case it holds that $b_{min} > \underline{b}_j = v_j$. Thus, if $v_i < v_j$ we get a contradiction as $b_{min} - v_i > v_j - v_i$ and $v_j - v_i$ is some positive constant (bounded away from 0). We conclude that $b_{min} = \underline{b}_1 > \underline{b} = \underline{b}_2 = v_2$, that is, bidder 1 has an atom at $b_{min} = \underline{b}_1 > v_1$ and bidder 2 has an atom at $v_2 = \underline{b}_2 = \underline{b} < b_{min}$, as we need to show. □

Claim 4. *Assume $b_{max} > b_{min}$ and that ϵ is small enough. Assume that $Pr[H_1, L_2](1 - v_1) < Pr[L_1, H_2](1 - v_2)$. Then either bidder 1 has no atom and bidder 2 has an atom at $v_2 = \underline{b}_2 = \underline{b} = b_{min}$. Or, bidder 1 has an atom at $b_{min} = \underline{b}_1 > v_1$ and bidder 2 has an atom at $v_2 = \underline{b}_2 = \underline{b} < b_{min}$.*

Proof. By Claim 2 as bidders are not symmetric it cannot be the case that both bidders have no atom. We next consider the case that at least one bidder has an atom. By Lemma 62 b_{max} tends to 1 as ϵ goes to 0. Additionally, $T(b, b_{min})$ tends to 1 as b tends to 1 (by Lemma 64). Thus, the RHS of Equation (47) tends to $\chi = \frac{Pr[H_1, L_2](1 - v_1)}{Pr[L_1, H_2](1 - v_2)} < 1$ as ϵ goes to 0. Equation (47) combined with $\chi < 1$ implies that $G_1(b_{min}) < G_2(b_{min})$.

Now, if only one bidder has an atom it must be bidder 2, since $G_2(b_{min}) = 0$ implies $G_1(b_{min}) < 0$, a contradiction. If on the other hand both bidders have an atom, we claim that bidder 1 has an atom at $b_{min} = \underline{b}_1$ and bidder 2 has an atom at $v_2 = \underline{b}_2 = \underline{b} < b_{min}$. Observe also that $\beta(b_{max}) = \frac{\hat{R}(b_{min})}{\hat{R}(b_{max})}$ tends to 1 as ϵ goes to 0. Now, if bidder 2 is the bidder with the atom at b_{min}, by Equation (42) $G_2(b_{min})$ must tend to 0 as ϵ goes to 0 (as b_{min} does not tend to 1 the denominator does not tend to 0, while the numerator tends to 0). Combining with $G_1(b_{min}) < G_2(b_{min})$ this will imply that $G_1(b_{min})$ must also tend to 0 as ϵ goes to zero. But then the LHS of Equation (47) tends to 1 while the RHS tends to $\chi < 1$, a contradiction. We conclude that bidder 1 has an atom at $b_{min} = \underline{b}_1$ and bidder 2 has an atom at $v_2 = \underline{b}_2 = \underline{b} < b_{min}$. □

By Equation (47) $G_2(b_{min})$ must satisfy

$$G_2(b_{min}) = \frac{1}{\beta(b_{max})} - \left(\frac{1}{\beta(b_{max})} - G_1(b_{min})\right) \cdot \frac{Pr[H_1, L_2]}{Pr[L_1, H_2]} \cdot \frac{1 - v_1 T(b_{max}, b_{min})}{1 - v_2 T(b_{max}, b_{min})} \tag{50}$$

Now Equation (43) follows from the definition of $\beta(b_{max})$ and $T(b_{max}, b_{min})$. The other claims in the lemma for the case that bidder 1 has an atom at $b_{min} = \underline{b}_1$ and bidder 2 has an atom at $v_2 = \underline{b}_2 = \underline{b} < b_{min}$ directly follow from Lemma 20, Lemma 60 and Lemma 61.

Note that by Lemma 64 and the above observations, as ϵ goes to 0, the size of the atom $G_2(v_2)$ tends to $1 - \frac{Pr[H_1,L_2](1-v_1)}{Pr[L_1,H_2](1-v_2)}$. □

Lemma 64. *Fix any $0 \le b_{min} < 1$ and a standard distribution R with density r. The function*

$$T(b) = \frac{\int_{b_{min}}^{b} \frac{1}{1-x} \cdot r(x)dx}{\int_{b_{min}}^{b} \frac{x}{1-x} \cdot r(x)dx} \tag{51}$$

monotonically decreases to 1 as b increases from b_{min} to 1.

Additionally,

$$\frac{\int_{b_{min}}^{b} \frac{x-v_1}{1-x} r(x)dx}{\int_{b_{min}}^{b} \frac{x-v_2}{1-x} r(x)dx} = \frac{1 - v_1 \cdot T(b)}{1 - v_2 \cdot T(b)} \tag{52}$$

tends to $\frac{1-v_1}{1-v_2}$ as b tends to 1. If $v_1 > v_2$ it is monotonically increasing to its limit, and if $v_1 < v_2$ it is monotonically decreasing to its limit.

Proof. Let $c \ge b_{min}$ be some number such that $0 < c < 1$ (say, $c = b_{min}$ unless $b_{min} = 0$, in this case $c = 1/2$). Assume $b \ge c$. Since r is continuous on a compact set its infimum is obtained. Since r is positive for every x, $\exists \underline{r} > 0$ such that $r(x) \ge \underline{r}$ for every x. Then

$$\int_{b_{min}}^{b} \frac{1}{1-x} r(x)dx \ge \int_{b_{min}}^{b} \frac{x}{1-x} r(x)dx \ge \underline{r} \cdot \int_{c}^{b} \frac{x}{1-x} dx \ge c \cdot \underline{r} \cdot \int_{c}^{b} \frac{1}{1-x} dx$$

Now we observe that both the numerator and the denominator of $T(b)$ tend to infinity when b tends to 1 as

$$\lim_{b \to 1} \int_{c}^{b} \frac{1}{1-x} dx = \lim_{b \to 1} \left(\ln(1-c) - \ln(1-b)\right) = \infty$$

Thus by L'Hôpital's rule,

$$\lim_{b \to 1} \frac{\int_{b_{min}}^{b} \frac{1}{1-x} r(x)dx}{\int_{b_{min}}^{b} \frac{x}{1-x} r(x)dx} = \lim_{b \to 1} \frac{\frac{d}{db}\int_{b_{min}}^{b} \frac{1}{1-x} r(x)dx}{\frac{d}{db}\int_{b_{min}}^{b} \frac{x}{1-x} r(x)dx} = \lim_{b \to 1} \frac{\frac{1}{1-b} r(b)}{\frac{b}{1-b} r(b)} = \lim_{b \to 1} \frac{1}{b} = 1.$$

Next we show that $T(b)$ monotonically decreases to 1 as b increases to 1. For any $b < 1$ all terms are finite, so we can compute the derivative:

$$\frac{d}{db}\frac{\int_{b_{min}}^{b}\frac{1}{1-x}r(x)dx}{\int_{b_{min}}^{b}\frac{x}{1-x}r(x)dx} = \frac{1}{1-b}r\left(b\right)\frac{\int_{b_{min}}^{b}\frac{x-b}{1-x}r(x)dx}{\left(\int_{b_{min}}^{b}\frac{x}{1-x}r\left(x\right)dx\right)^{2}} < 0.$$

For $b < 1$, $\frac{1}{1-b} > 0$. For $0 \leq b_{min} < b < 1$ and $x \in [b_{min}, b]$, $\frac{x-b}{1-x} < 1$. Therefore $T(b)$ is monotonically decreasing to 1 as b increases to 1.

Observe that

$$\frac{\int_{b_{min}}^{b}\frac{x-v_1}{1-x}r(x)dx}{\int_{b_{min}}^{b}\frac{x-v_2}{1-x}r(x)dx} = \frac{1-v_1\cdot T(b)}{1-v_2\cdot T(b)} = 1-\frac{v_1-v_2}{1/T(b)-v_2}. \tag{53}$$

When $v_1 > v_2$ it is monotonically increasing to $\frac{1-v_1}{1-v_2}$ as b increases to 1, since $T(b)$ decreases to 1 and $v_1 - v_2 > 0$. Similar argument shows that when $v_1 < v_2$ it is monotonically decreasing to its limit. □

D.2.3 Convergence to the TRE

For a standard distribution R it holds that its density function r is bounded as r is a continuous function on a compact set. Thus there exists some bound $r_{max} < \infty$ such that $r_{max} \geq r(x)$ for all x.

Lemma 65. *If ϵ is small enough then the following holds. For every $b \in (b_{min}, b_{max})$ in the support of G_1 it must holds that:*

$$\frac{Pr[L_1|H_2]}{Pr[H_1|H_2]}\cdot\frac{\epsilon}{1-\epsilon}\cdot r_{max}\cdot(-b-log(1-b))+G_1(b_{min}) \geq G_1(b) \tag{54}$$

where for r_{max} it holds that $r_{max} \geq r(x)$ for all x.

Proof. By Lemma 60

$$G_1(b) = \frac{Pr[L_1|H_2]}{Pr[H_1|H_2]}\cdot\frac{\epsilon}{\hat{R}(b)}\cdot\int_{b_{min}}^{b}\frac{x-v_2}{1-x}r(x)dx + G_1(b_{min})\cdot\frac{\hat{R}(b_{min})}{\hat{R}(b)} \tag{55}$$

As $v_2 \geq 0$ and $r(b) \leq r_{max}$ for all b,

$$\int_{b_{min}}^{b}\frac{x-v_2}{1-x}r(x)dx \leq \int_{b_{min}}^{b}\frac{x}{1-x}r(x)dx \leq r_{max}\int_{0}^{b}\frac{x}{1-x}dx = r_{max}(-b-log(1-b)) \tag{56}$$

As $\hat{R}(b) \geq \hat{R}(b_{min}) \geq 1-\epsilon$ we conclude that

$$G_1(b) \leq \frac{Pr[L_1|H_2]}{Pr[H_1|H_2]}\cdot\frac{\epsilon}{1-\epsilon}\cdot r_{max}(-b-log(1-b))+G_1(b_{min}) \tag{57}$$

□

Lemma 66. *If ϵ is small enough then the following holds. For every $b \in (b_{min}, b_{max})$ in the support of G_2 it must holds that:*

$$\frac{Pr[L_2|H_1]}{Pr[H_2|H_1]}\cdot\frac{\epsilon}{1-\epsilon}\cdot r_{max}\cdot(-b-log(1-b)) \geq G_2(b)-G_2(v_2) \tag{58}$$

where for r_{max} it holds that $r_{max} \geq r(x)$ for all x.

Proof. The proof is the same as the proof of Lemma 65 when using the following equation proved in Lemma 61 and the fact that $G_2(b_{min}) = G_2(v_2)$:

$$G_2(b) = \frac{Pr[L_2|H_1]}{Pr[H_2|H_1]} \cdot \frac{\epsilon}{\hat{R}(b)} \int_{b_{min}}^{b} \frac{x - v_1}{1 - x} \cdot r(x)dx + G_2(b_{min}) \cdot \frac{\hat{R}(b_{min})}{\hat{R}(b)} \tag{59}$$

□

Lemma 21, Lemma 65 and Lemma 66 enable us to prove that any TRE must be μ.

Corollary 67. *Fix a standard distribution R. For every $b \in [0, \min\{v_2, v_1\})$ it holds that $G_1(b) = G_2(b) = 0$. For every $b \in [\min\{v_2, v_1\}, 1)$ the limits of $G_1(b)$ and of $G_2(b) - G_2(v_2)$, as ϵ goes to zero, are both zero. Additionally, $G_2(v_2)$ tends to $1 - \frac{Pr[H_1,L_2](1-v_1)}{Pr[L_1,H_2](1-v_2)}$ as ϵ goes to zero.*

Proof. Fix some $b < 1$. For a small $\delta > 0$ such that $\delta < 1 - b$, by Lemma 53 and Lemma 62 it holds that for a small enough ϵ, $b_{max} > 1 - \delta$ and $b_{max} > b_{min}$. Now it holds that $b < b_{max}$.

Since $\underline{b}_1 \geq v_1$ and $\underline{b}_2 \geq v_2$ it holds that $\underline{b} = \min\{\underline{b}_1, \underline{b}_2\} \geq \min\{v_1, v_2\}$, thus by Lemma 21 $G_1(b) = G_2(b) = 0$ for every $b \in [0, \min\{v_1, v_2\})$. The same lemma also implies that $G_1(b) = G_2(b) - G_2(v_2) = 0$ for every $b \in [\min\{v_1, v_2\}, b_{min})$.

Next we consider $b \in [b_{min}, 1)$. We first observe that for any fixed $b \in (b_{min}, 1)$, by Lemma 65, $G_1(b)$ tends to $G_1(b_{min})$ as ϵ goes to 0 (and clearly $G_1(b) = G_1(b_{min})$ for $b = b_{min}$). The claim that $G_1(b)$ tends to 0 follows from Lemma 21 which shows that $G_1(b_{min})$ is either 0 or tends to 0 when ϵ goes to 0. Additionally, by Lemma 66, for any fixed $b \in (b_{min}, 1)$ it holds that $G_2(b)$ tends to $G_2(v_2)$ as ϵ goes to 0. As G_2 is continuous at b_{min} the claim also hold at that point.

Finally, Lemma 21 combined with Lemma 64 show that $G_2(v_2)$ is 0 if and and only if the bidders are symmetric and $\frac{Pr[H_1,L_2](1-v_1)}{Pr[L_1,H_2](1-v_2)} = 1$, and otherwise $G_2(v_2)$ tends to $1 - \frac{Pr[H_1,L_2](1-v_1)}{Pr[L_1,H_2](1-v_2)}$ as ϵ goes to zero. □

D.2.4 Existence of NE in $\lambda(\epsilon, R)$

Observe that $R(b) = b$ is a standard distribution, so standard distributions exist. We next show that for any standard distribution R, if ϵ is small enough then there exists a mixed NE in the game $\lambda(\epsilon, R)$.

We prove existence of one of three types of equilibria depending on parameter values. For symmetric bidders, we show the existence of an equilibrium with no atoms (case 1). For asymmetric bidders we show the existence of either a one-atom (case 2) or a two-atom (case 3) equilibrium depending on whether or not equation (63) in the proof is satisfied. The following observation indicates why equation (63) determines whether asymmetric equilibria involve one or two atoms.

Observation 68. *If ϵ is small enough and $G_1(b_{min}) > 0$ (bidder 1 has an atom, which implies that bidder 2 also has an atom) then it must hold that*

$$\alpha_2 \cdot \frac{v_2 - v_1}{1 - v_2} \leq 1 - \frac{\alpha_2}{\alpha_1} \cdot \frac{1 - v_1}{1 - v_2} \tag{60}$$

Proof. If $G_1(b_{min}) > 0$ then Equation (41) holds. In particular it must holds that

$$\frac{G_2(v_2) + v_1\alpha_2}{G_2(v_2) + \alpha_2} = 1 - \frac{\alpha_2(1 - v_1)}{G_2(v_2) + \alpha_2} > v_2 \tag{61}$$

By Corollary 67, $G_2(v_2)$ tends to $1 - \frac{Pr[H_1,L_2](1-v_1)}{Pr[L_1,H_2](1-v_2)} = 1 - \frac{\alpha_2(1-v_1)}{\alpha_1(1-v_2)}$ as ϵ goes to zero. Thus it must holds that

$$1 - \frac{\alpha_2(1-v_1)}{\left(1 - \frac{\alpha_2(1-v_1)}{\alpha_1(1-v_2)}\right) + \alpha_2} \geq v_2 \tag{62}$$

and the claim follows from reorganizing the last equation. □

Lemma 69. *Fix any standard distribution R. For every small enough $\epsilon > 0$ there exists a mixed NE η in the game $\lambda(\epsilon, R)$.*

Proof. Let $\bar{v} = \max\{v_1, v_2\}$. Throughout the proof we index bidders 1 and 2 such that either 1) $\alpha_1(1-v_2) = \alpha_2(1-v_1)$ and $v_1 > v_2$, or 2) $\alpha_1(1-v_2) > \alpha_2(1-v_1)$. Moreover, we often distinguish between three cases:

1. *No atom case.* Bidders are symmetric: $v = v_1 = v_2$ and $Pr[H_1, L_2] = Pr[L_1, H_2]$. In this case we show there exists an equilibrium in which $b_{min} = v$ and neither bidder has an atom: $G_1(b_{min}) = G_2(v_2) = 0$.
2. *One atom case.* Bidders are asymmetric ($v_1 \neq v_2$ or $Pr[H_1, L_2] \neq Pr[L_1, H_2]$) and equation (63) holds:
$$\alpha_2 \cdot \frac{v_2 - v_1}{1 - v_2} \geq 1 - \frac{\alpha_2}{\alpha_1} \cdot \frac{1 - v_1}{1 - v_2}. \tag{63}$$
Note that asymmetry and equation (63) imply that $\alpha_1(1-v_2) > \alpha_2(1-v_1)$ and $v_2 > v_1$. This is so as by assumption the RHS of equation (63) is non-negative, this implies that $v_2 \geq v_1$. If $v_2 = v_1$ Then the equation implies that $\alpha_1 = \alpha_2$ which means the bidders are symmetric, a contradiction. Therefore $v_2 > v_1$ and thus $\alpha_1(1-v_2) > \alpha_2(1-v_1)$ (since in the case that $\alpha_1(1-v_2) = \alpha_2(1-v_1)$ we assume that $v_1 > v_2$).
In this case we show that there exists an equilibrium in which $b_{min} = v_2$ and only bidder 2 has an atom: $G_2(v_2) > 0$ and $G_1(b_{min}) = 0$.
3. *Two atom case.* Bidders are asymmetric ($v_1 \neq v_2$ or $Pr[H_1, L_2] \neq Pr[L_1, H_2]$) and equation (63) is violated. Note that either 1) $\alpha_1(1-v_2) = \alpha_2(1-v_1)$ and $v_1 > v_2$, or 2) $\alpha_1(1-v_2) > \alpha_2(1-v_1)$ are both feasible. In this case we show that there exists an equilibrium in which $b_{min} > \max\{v_1, v_2\}$ and both bidders have atoms: $G_2(v_2) > 0$ and $G_1(b_{min}) > 0$.

In all cases, bidder i with signal L_i is bidding $v(L_1, L_2) = 0$. We construct distributions G_1 and G_2 using the necessary conditions in Lemma 21 and show that they form a NE. Equations (44) and (45) define G_1 and G_2 as a function of the four parameters b_{min}, b_{max}, $G_1(b_{min})$, and $G_2(v_2)$. There are three main steps to the proof. First we show existence of parameters b_{min}, b_{max}, $G_1(b_{min})$, and $G_2(v_2)$ that satisfy the necessary conditions in Lemma 21. Second, we show that, for the chosen parameters, G_1 and G_2 are well defined distributions (non-decreasing, and satisfying $G_1(0) = G_2(0) = 0$ and $G_1(1) = G_2(1) = 1$). Third we show that the constructed bid distributions are best responses. By construction, bidder $i \in \{1, 2\}$ is indifferent to all bids in the support of his bid distribution and we show that every bid outside the support gives weakly lower utility.

Step 1. Existence of parameters b_{min}, b_{max}, $G_1(b_{min})$, and $G_2(v_2)$:

Case 1 (no atoms): First consider the case that the bidders are symmetric. We define $b_{min} = v$ and $G_1(b_{min}) = G_2(v_2) = 0$. By the necessary conditions at b_{max} it must hold that

$$1 = G_1(b_{max}) = \frac{Pr[L_1|H_2]}{Pr[H_1|H_2]} \cdot \frac{\epsilon}{\hat{R}(b_{max})} \cdot \int_v^{b_{max}} \frac{x - v}{1 - x} r(x) dx \tag{64}$$

The RHS increases from zero to infinity as b_{max} increases from v to 1 (Claim 6), so there exists a unique value of $b_{max} \in (v, 1)$ that solves this equation. It is clear that b_{max} must tend to 1 as ϵ goes to 0. Note that all the necessary conditions presented in Lemma 21 for the symmetric case are now satisfied.

Case 2 (one atom): Next consider the case that bidders are asymmetric and equation (63) holds (implying $\alpha_2 \cdot (1 - v_1) < \alpha_1 \cdot (1 - v_2)$ and $v_1 < v_2$). We define $b_{min} = v_2$ and $G_1(b_{min}) = 0$. As $G_1(b_{min}) = 0$, $b_{max} \in (v, 1)$ can be determined exactly as in the symmetric case. Finally, we set $G_2(v_2)$ using Equation (43). Observe that $G_2(v_2)$ as defined tends to $1 - \frac{\alpha_2}{\alpha_1} \cdot \frac{1-v_1}{1-v_2} \in (0, 1)$ as ϵ tends to 0, thus for sufficiently small ϵ it is positive.

Case 3 (two atoms): Finally, consider the case that bidders are asymmetric and equation (63) is violated. We define $G_1(b_{min})$ as a function of b_{min} and b_{max} by equation (42). We define $G_2(v_2)$ as a function of b_{min} by equation (41), or equivalently by:

$$G_2(v_2) = \frac{\Pr[L_2|H_1]}{\Pr[H_2|H_1]} \frac{b_{min} - v_1}{1 - b_{min}}. \tag{65}$$

The arguments below show that $b_{min} > \max\{v_1, v_2\}$, which ensures that $G_1(b_{min}) > 0$ and $G_2(v_2) > 0$. By substituting $G_1(b_{min})$ and $G_2(v_2)$ into equations (44) and (45), which determine $G_1(b)$ and $G_2(b)$, and evaluating these equations at b_{max}, for which it must hold that $G_1(b_{max}) = G_2(b_{max}) = 1$, we derive that we need to find b_{min} and b_{max} that satisfy the following pair of equations:

$$1 = \alpha_1 \cdot \frac{1}{\hat{R}(b_{max})} \int_{b_{min}}^{b_{max}} \frac{x - v_2}{1 - x} \cdot \hat{r}(x)dx + \alpha_1 \cdot \frac{1}{\hat{R}(b_{max})} \int_{v_2}^{b_{min}} \frac{x - v_2}{1 - b_{min}} \cdot \hat{r}(x)dx \tag{66}$$

$$1 = \alpha_2 \cdot \frac{1}{\hat{R}(b_{max})} \int_{b_{min}}^{b_{max}} \frac{x - v_1}{1 - x} \cdot \hat{r}(x)dx + \frac{b_{min} - v_1}{1 - b_{min}} \cdot \alpha_2 \cdot \frac{\hat{R}(b_{min})}{\hat{R}(b_{max})} \tag{67}$$

We first show that when ϵ is small enough, for any $b_{min} \in [\bar{v}, v(H_1)]$ we can find a unique $b_{max} \in (b_{min}, 1)$ that solves equation (66). We denote such a solution by $b_{max}(b_{min})$. When $b_{max} = b_{min}$, the RHS of equation (66) equals $\epsilon \cdot h(b_{min})$ for $h(b_{min}) = \frac{\alpha_1}{\hat{R}(b_{min})} \int_{v_2}^{b_{min}} \frac{x-v_2}{1-b_{min}} \cdot r(x)dx$. As h is a continuous function on a compact set it is bounded, thus $\epsilon \cdot h(b_{min}) < 1$ for any $b_{min} \in [\bar{v}, v(H_1)]$ as long as ϵ is small enough. Now, for every fixed $b_{min} \in [\bar{v}, v(H_1)]$, the RHS of equation (66) is continuously increasing in b_{max} (by Claim 6 below) and goes to infinity when b_{max} tends to 1. Therefore there exists a unique $b_{max} \in (b_{min}, 1)$ that solves the equation. Note that $b_{max}(b_{min})$ is a continuous function of b_{min} and, for any fixed b_{min}, $b_{max}(b_{min})$ tends to 1 as ϵ tends to 0.

Now we substitute $b_{max}(b_{min})$ into equation (67) and get the following equation in b_{min}

$$1 = \alpha_2 \cdot \frac{1}{\hat{R}(b_{max}(b_{min}))} \int_{b_{min}}^{b_{max}(b_{min})} \frac{x - v_1}{1 - x} \cdot \hat{r}(x)dx + \frac{b_{min} - v_1}{1 - b_{min}} \cdot \alpha_2 \cdot \frac{\hat{R}(b_{min})}{\hat{R}(b_{max}(b_{min}))} \tag{68}$$

To complete the proof we need to show that there exists $b_{min} \in [\bar{v}, v(H_1)]$ that satisfies equation (68). The RHS of this equation is a continuous function of b_{min} on the compact set $[\bar{v}, v(H_1)]$. It will therefore be sufficient to show that for $b_{min} = v(H_1)$ the RHS is strictly larger than 1, while for $b_{min} = \bar{v}$ the RHS is strictly smaller than 1. Once this is shown (below) we conclude that there exists $b_{min} > \bar{v}$ such that the RHS is exactly 1. This b_{min} together with $b_{max} = b_{max}(b_{min})$ solve both equations (66) and (67) and satisfy $1 > b_{max} > b_{min} > \bar{v}$.

To prove the remaining two inequalities, define:

$$z(b_{min}) = \alpha_1 \cdot \frac{1}{\hat{R}(b_{max}(b_{min}))} \int_{b_{min}}^{b_{max}(b_{min})} \frac{x - v_2}{1 - x} \cdot \hat{r}(x)dx.$$

Now, the RHS of equation (68) can be written as

$$z(b_{min}) \cdot \frac{\alpha_2 \cdot \int_{b_{min}}^{b_{max}(b_{min})} \frac{x-v_1}{1-x} \cdot \hat{r}(x)dx}{\alpha_1 \cdot \int_{b_{min}}^{b_{max}(b_{min})} \frac{x-v_2}{1-x} \cdot \hat{r}(x)dx} + \frac{b_{min} - v_1}{1 - b_{min}} \cdot \alpha_2 \cdot \frac{\hat{R}(b_{min})}{\hat{R}(b_{max}(b_{min}))} \tag{69}$$

Fix b_{min}. Note that equation (66) implies that $z(b_{min}) \leq 1$ and $z(b_{min})$ tends to 1 as ϵ goes to 0, as the second term of the RHS of equation (66) is positive and tends to 0. By Lemma 64, $\frac{\alpha_2 \cdot \int_{b_{min}}^{b_{max}(b_{min})} \frac{x-v_1}{1-x} \cdot \hat{r}(x)dx}{\alpha_1 \cdot \int_{b_{min}}^{b_{max}(b_{min})} \frac{x-v_2}{1-x} \cdot \hat{r}(x)dx}$ tends to $\frac{\alpha_2(1-v_1)}{\alpha_1(1-v_2)}$, and thus, as ϵ tends to 0, the RHS of equation (68) tends to

$$\frac{\alpha_2(1 - v_1)}{\alpha_1(1 - v_2)} + \frac{b_{min} - v_1}{1 - b_{min}} \cdot \alpha_2 \tag{70}$$

For $b_{min} = v(H_1)$, equation (70) strictly exceeds 1 since by equation (41) it holds that $b_{min} = v(H_1)$ if and only if $G_2(v_2) = \frac{b_{min}-v_1}{1-b_{min}} \cdot \alpha_2 = 1$, and the first term is strictly positive by assumption. Thus, for sufficiently small ϵ, the RHS of equation (68) also strictly exceeds 1 for $b_{min} = v(H_1)$.

If $b_{min} = \bar{v}$ we show that the RHS of equation (68) is strictly less than 1 for sufficiently small ϵ. We consider two cases separately. First, if $b_{min} = \bar{v} = v_2 \geq v_1$, equation (70) is strictly less than 1 as equation (63) is violated. Thus, for sufficiently small ϵ, the RHS of equation (68) is also strictly less than 1. Second, if $b_{min} = \bar{v} = v_1 > v_2$, equation (70) is weakly (but not necessarily strictly) less than 1. However, we show that equation (69) (and hence the RHS of equation (68)) is strictly less than equation (70) for all $\epsilon > 0$. This follows because $b_{min} > v_2$ implies that the second term on the RHS of equation (66) is strictly positive so that $z(b_{min}) < 1$ and $v_1 > v_2$ implies (by Lemma 64) that $\frac{\alpha_2 \cdot \int_{b_{min}}^{b_{max}} \frac{x-v_1}{1-x} \cdot \hat{r}(x)dx}{\alpha_1 \cdot \int_{b_{min}}^{b_{max}} \frac{x-v_2}{1-x} \cdot \hat{r}(x)dx}$ is increasing to its limit (which is at most 1).

Step 2. G_1 and G_2 are well defined: We next argue that G_1 and G_2, as defined above by Step 1 and equations (44) and (45), are well defined distributions. The way we have chosen the parameters in Step 1 ensures that $\max\{v_1, v_2\} \leq b_{min} < b_{max} \leq 1$, $G_1(b_{min}), G_2(v_2) \geq 0$, and $G_1(b_{max}) = G_2(b_{max}) = 1$. The two distributions are continuous from the right at b_{min}, and by Claim 6 and Claim 5 are strictly increasing on (b_{min}, b_{max}). Thus both are monotonically non-decreasing on $[0, \infty)$ with $G_1(0) = G_2(0) = 0$ and $G_1(b_{max}) = G_2(b_{max}) = 1$.

Step 3. Constructed bid distributions are best responses: To see that η is indeed a mixed NE we show that each bidder is best responding to the other. Observe that, by construction, G_1 and G_2 ensure that each bidder is indifferent between all the bids in the support her bid distribution. It only remains to show that all other bids earn weakly lower payoffs.

First consider bids above b_{max}. As $0 < Pr[H_1, L_2](1 - v(H_1, L_2)) \leq Pr[L_1, H_2](1 - v(L_1, H_2))$ it holds that $\max\{v(H_1), v(H_2)\} < 1$. Therefore, as b_{max} tends to 1 when ϵ tends to 0, for small enough ϵ it holds that $b_{max} > \max\{v(H_1), v(H_2)\}$. Therefore, for small enough ϵ, Lemma 43 implies that for both bidders b_{max} strictly dominates any higher bid $b > b_{max}$.

Second note that Lemma 43 also implies that for bidder i, bidding v_i strictly dominates any lower bid $b < v_i$.

Third, we consider bids $b \in [v_i, b_{min}]$ by bidder $i \in \{1, 2\}$ outside the support of bidder i's bid distribution for each of the three cases.

Consider case 1 (no atoms) in which $b_{min} = v_1 = v_2 = v$. In this case, the utility from bidding $b_{min} = v$ equals the utility of any bid in $[v, b_{max}]$ by continuity.

Consider case 2 (one atom) in which $b_{min} = v_2$, $\alpha_2 \cdot (1 - v_1) < \alpha_1 \cdot (1 - v_2)$, and $v_2 > v_1$. Bidder 2 bids an atom at v_2 so there are no other bids to check. For bidder 1, Lemma 40 implies that any bid in (v_2, b_{max}) strictly dominates bidding v_2. By Lemma 43, the bid with the highest payoff strictly below v_2 is v_1. By bidding v_1, bidder 1 never wins when bidder 2 gets the high signal H_2. Since $1 - \frac{\alpha_2}{\alpha_1} \cdot \frac{1-v_1}{1-v_2} > 0$ the size of the atom of bidder 2 does not tend to 0 as ϵ tends to 0, and clearly the gain by bidding above the atom of bidder 2 at v_2 instead of bidding v_1 is positive if ϵ is small enough.

Consider case 3 (two atoms) in which $b_{min} > \max\{v_1, v_2\}$. Bidder 2 bids an atom at v_2, which by Lemma 43 dominates any bid $b < b_{min}$. Moreover, for bidder 2, bidding b_{min} is dominated by bids in the support by Lemma 40. Now turn to bidder 1. Lemmas 43 and 40 imply that i's atom at b_{min} dominates any bid in $[v_2, b_{min})$ because b_{min} is defined by equation (41). For $v_1 \geq v_2$, $[v_2, b_{min})$ includes all bids $[v_1, b_{min})$ and we are done. For $v_1 < v_2$, we must also consider bids $[v_1, v_2)$, of which v_1 gives the highest payoff by Lemma 43. As $v_1 < v_2$ implies $\alpha_2 \cdot (1 - v_1) < \alpha_1 \cdot (1 - v_2)$, b_{min} must dominate v_1 for sufficiently small ϵ by the same argument applied above in the one-atom case.

Claim 5. *In all three cases (no atoms, one atom, two atoms) $G_2(b)$ as defined above is increasing in b for every $b \in (b_{min}, b_{max})$.*

Proof. We need to show that in all three cases $G_2(b)$ is increasing in b for every $b \in (b_{min}, b_{max})$. For any such b, $G_2(b)$ satisfies Equation (39), and its derivative with respect to b is

$$g_2(b) = \frac{\hat{r}(b)}{\hat{R}(b)} \left(\alpha_2 \cdot \frac{b - v_1}{1 - b} - G_2(b) \right).$$

To prove the claim it is sufficient to show that for every $b \in (b_{min}, b_{max})$:

$$g_2(b) \cdot \frac{\hat{R}(b)}{\hat{r}(b)} = \alpha_2 \cdot \frac{b - v_1}{1 - b} - G_2(b) > 0. \tag{71}$$

If $G_2(b) \leq 0$ the claim follows from $1 \geq b_{max} > b > b_{min} \geq \max\{v_1, v_2\}$. Next assume that $G_2(b) \geq 0$. We observe that for small enough ϵ this is an increasing function in b for $b \in (b_{min}, b_{max})$:

$$\begin{aligned} \frac{d}{db}\left(\frac{\hat{R}(b)}{\hat{r}(b)} g_2(b)\right) &= \alpha_2 \frac{1 - v_1}{(1 - b)^2} - g_2(b) = \alpha_2 \frac{1 - v_1}{(1 - b)^2} - \frac{\hat{r}(b)}{\hat{R}(b)} \left(\alpha_2 \frac{b - v_1}{1 - b} - G_2(b) \right) \\ &\geq \alpha_2 \frac{1}{(1 - b)^2} \left((1 - v_1) - \frac{\hat{r}(b)}{\hat{R}(b)} (b - v_1)(1 - b) \right) \\ &\geq \alpha_2 \frac{1}{(1 - b_{min})^2} \left(1 - v_1 - \varepsilon \frac{r(b)}{1 - \epsilon} \right). \end{aligned}$$

As $1 > v_1$ and $r(b)$ is bounded from above (r is continuous on a compact interval), for small enough ϵ this is positive.

Thus, as the function $\frac{\hat{R}(b)}{\hat{r}(b)}g_2(b)$ is increasing, to prove that it is positive for any $b > b_{min}$ it would be sufficient to show that it is at least 0 at b_{min}, or equivalently, that the following holds:

$$\alpha_2 \cdot \frac{b_{min} - v_1}{1 - b_{min}} \geq G_2(b_{min}). \tag{72}$$

We show that equation (72) is satisfied for each of the three cases.

In the first case (no atoms), $G_2(v_2) = 0$, and equation (72) clearly holds because $b_{min} \geq v_1$. In the third case (two atoms), $G_2(v_2)$ satisfies equation (41), which is exactly equivalent to equation (72) holding with equality.

Finally we consider the second case (one atom) in which $\alpha_2 \cdot (1 - v_1) < \alpha_1 \cdot (1 - v_2)$, equation (63) holds and $G_2(b_{min}) = G_2(v_2) > 0$ satisfies equation (43) with $G_1(b_{min}) = 0$, and additionally, $b_{min} = v_2 > v_1$ (this corresponds to the case that only bidder 2 has an atom). These conditions imply that

$$G_2(v_2) = \frac{\hat{R}\left(b_{max}\right)}{\hat{R}\left(v_2\right)}\left(1 - \frac{\alpha_2 \int_{v_2}^{b_{max}} \frac{x - v_1}{1-x} r(x)dx}{\alpha_1 \int_{v_2}^{b_{max}} \frac{x-v_2}{1-x} r(x)dx}\right).$$

Which means that we need to show that

$$\alpha_2 \frac{v_2 - v_1}{1 - v_2} \geq \frac{\hat{R}\left(b_{max}\right)}{\hat{R}\left(v_2\right)}\left(1 - \frac{\alpha_2 \int_{v_2}^{b_{max}} \frac{x - v_1}{1-x} r(x)dx}{\alpha_1 \int_{v_2}^{b_{max}} \frac{x-v_2}{1-x} r(x)dx}\right) = G_2(v_2)$$

Equation (64) determines b_{max} and implies that $\hat{R}(b_{max}) = \alpha_1 \int_{v_2}^{b_{\max}} \frac{x-v_2}{1-x}\hat{r}(x)dx$, thus:

$$\frac{\hat{R}(b_{max})}{\hat{R}(v_2)} = \frac{\hat{R}(b_{max})}{\hat{R}(b_{max}) - \int_{v_2}^{b_{\max}} \hat{r}(x)dx} = \frac{\alpha_1 \int_{v_2}^{b_{\max}} \frac{x-v_2}{1-x} r(x)dx}{\int_{v_2}^{b_{\max}} \left(\alpha_1 \frac{x-v_2}{1-x} - 1\right) r(x)dx}$$

We can now express $G_2(v_2)$ as a function of b_{max} as follows:

$$\begin{aligned} G_2(v_2) &= \frac{\alpha_1 \int_{v_2}^{b_{\max}} \frac{x-v_2}{1-x} r(x)dx}{\int_{v_2}^{b_{\max}} \left(\alpha_1 \frac{x-v_2}{1-x} - 1\right) r(x)dx}\left(1 - \frac{\alpha_2 \int_{v_2}^{b_{\max}} \frac{x-v_1}{1-x} r(x)dx}{\alpha_1 \int_{v_2}^{b_{\max}} \frac{x-v_2}{1-x} r(x)dx}\right) \\ &= \frac{\int_{v_2}^{b_{\max}} \left(\alpha_1 \left(x - v_2\right) - \alpha_2 \left(x - v_1\right)\right) \frac{r(x)}{1-x} dx}{\int_{v_2}^{b_{\max}} \left(\alpha_1 \frac{x-v_2}{1-x} - 1\right) r(x)dx} \end{aligned}$$

b_{max} tends to 1 as ϵ goes to 0 (Lemma 62) and $G_2(v_2)$ tends to $1 - \frac{\alpha_2}{\alpha_1} \cdot \frac{1-v_1}{1-v_2}$ as b_{max} tends to 1 (Corollary 67). By Equation (63) it is thus sufficient to prove that $G_2(v_2)$ is nondecreasing in $b_{\max}$: $\frac{d}{db_{\max}} G_2\left(v_2\right) \geq 0$.

$$
\begin{aligned}
\frac{dG_2(v_2)}{db_{\max}} &= \frac{1}{\left(\int_{v_2}^{b_{\max}}\left(\alpha_1\frac{x-v_2}{1-x}-1\right)r(x)dx\right)^2}\cdot\frac{r(b_{max})}{1-b_{\max}} \\
&\cdot\left(\begin{array}{c}\left(\alpha_1\left(b_{\max}-v_2\right)-\alpha_2\left(b_{\max}-v_1\right)\right)\int_{v_2}^{b_{\max}}\left(\alpha_1\left(x-v_2\right)-\left(1-x\right)\right)\frac{r(x)}{1-x}dx \\ -\left(\alpha_1\left(b_{\max}-v_2\right)-\left(1-b_{\max}\right)\right)\int_{v_2}^{b_{\max}}\left(\alpha_1\left(x-v_2\right)-\alpha_2\left(x-v_1\right)\right)\frac{r(x)}{1-x}dx\end{array}\right) \\
&= \frac{1}{\left(\int_{v_2}^{b_{\max}}\left(\alpha_1\frac{x-v_2}{1-x}-1\right)r(x)dx\right)^2}\cdot\frac{r(b_{max})}{1-b_{\max}} \\
&\cdot\int_{v_2}^{b_{\max}}\frac{b_{\max}-x}{1-x}\left(\alpha_1\alpha_2\left(v_2-v_1\right)-\alpha_1\left(1-v_2\right)+\alpha_2\left(1-v_2\right)\right)r(x)dx \\
&= \alpha_1\left(1-v_2\right)\left(\alpha_2\frac{v_2-v_1}{1-v_2}-\left(1-\frac{\alpha_2\left(1-v_2\right)}{\alpha_1\left(1-v_2\right)}\right)\right)\frac{\frac{r(b_{max})}{1-b_{\max}}\int_{v_2}^{b_{\max}}\frac{b_{\max}-x}{1-x}r(x)dx}{\left(\int_{v_2}^{b_{\max}}\left(\alpha_1\frac{x-v_2}{1-x}-1\right)r(x)dx\right)^2}
\end{aligned}
$$

By Equation (63), $\alpha_2\frac{v_2-v_1}{1-v_2}\geq\left(1-\frac{\alpha_2(1-v_2)}{\alpha_1(1-v_2)}\right)$, thus $\frac{dG_2(v_2)}{db_{\max}}\geq 0$ holds. (Moreover, when $\alpha_2\frac{v_2-v_1}{1-v_2}=\left(1-\frac{\alpha_2(1-v_2)}{\alpha_1(1-v_2)}\right)$, $\frac{dG_2(v_2)}{db_{\max}}=0$ and $G_2\left(v_2\right)$ attains its limit for any $b_{\max}<1$). □

Claim 6. *In all three cases (no atoms, one atom, two atoms) $G_1(b)$ as defined above is increasing in b for every $b\in(b_{min},b_{max})$.*

Proof. The same arguments as the ones presented in the proof of Claim 5 show that it is sufficient to prove that

$$\alpha_1\cdot\frac{b_{min}-v_2}{1-b_{min}}\geq G_1(b_{min}). \tag{73}$$

When bidder 1 does not have an atom (when no bidder has an atom, or only bidder 2 has an atom), this trivially holds since $b_{min}\geq v_2$. We are left to prove the claim when both bidders have an atom and $G_1(b_{min})>0$ satisfies Equation (42). We need to show that

$$\alpha_1\cdot\frac{b_{min}-v_2}{1-b_{min}}\geq\alpha_1\cdot\frac{\int_{v_2}^{b_{min}}\left(x-v_2\right)\hat{r}\left(x\right)dx}{\hat{R}\left(b_{min}\right)\left(1-b_{min}\right)}, \tag{74}$$

which trivially holds since $\hat{R}(b_{min})\geq\int_{v_2}^{b_{min}}\hat{r}\left(x\right)dx=\hat{R}(b_{min})-\hat{R}(v_2)$.

□

□