
Core-Selecting Auctions

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Package Allocation Problem

- Set of players N comprises
 - One single seller, player 0
 - Risk neutral bidders, quasi-linear preferences
 - X = set of feasible allocations
- Feasible allocations for a coalition S are $X_S = \{x \in X \mid x_j = 0 \text{ for } j \notin S\}$.
 - Bidder j 's value is $u_j(x_j) \geq 0$.
 - Coalition value is

$$w(S) = \begin{cases} 0 & \text{if } 0 \notin S \\ \max_{x \in X_S} \sum_{j \in S} u_j(x_j) & \text{otherwise} \end{cases}$$



Example: Procuring “Universal Service”

- US law mandates “basic communication services” be universally available.
- “High cost fund” adds 9% to every phone bill; much is wasted.
- Auctions are used in India and smaller countries to procure such services.
- Question: *how best to procure service guarantees in high cost areas when there are extensive cost interactions from shared infrastructure that vary by technology?*



Approach

- Incentives are just one of many important issues, not to be given absolute primacy.
- Trade-offs among incentives, revenues, and other properties are to be investigated.

Heresy for traditional economic analysis.



Reminder: The Core

- In game theory/economics, the core has two definitions:
 - The “core” is defined as a set of payoffs or “imputations,” as described below.
 - The “core” is the set of allocations whose imputed payoffs are core imputations.
- Given a coalitional game (N, w) ,

$$\text{Core}(N, w) = \left\{ \pi \geq 0 \mid \sum_{j \in N} \pi_j \leq w(N), (\forall S) \sum_{j \in S} \pi_j \geq w(S) \right\}$$

- This also defines a competitive equilibrium price vector π in a world with one consumption good where seller and bidders are “resources for hire” and two or more “firms” have identical production functions is $w(\cdot)$.



Why Not the Vickrey Auction?

- Example: 2 goods, 3 bidders
 - Bidder 1 will pay nothing for one item, 10 only for two items.
 - Bidders 2 & 3 will each pay 10 for one item or two.
 - Vickrey outcome: 2 & 3 win and pay 0!
- Example illustrates severe problems for Vickrey auctions
 - Vickrey revenues can be unacceptably low—even zero.
 - Vickrey outcome is not in the core—not “competitive.”
 - A merged player 2&3 would benefit by using two bidders.
 - Including bidder 3 reduces seller revenues.
 - Seller may wish to disqualify a real bidder 3.
 - Bidder 2 alone would wish to sponsor a shill to bid as 3.
 - More details in Ausubel & Milgrom (2005), “The Lovely But Lonely Vickrey Auction,” in *Combinatorial Auctions*, MIT Press.



A Class of Easy Cases

- Let M be any of the many direct mechanisms that select a *bidder optimal core allocation* using the bidders' reported values.
- If goods are substitutes for all bidders, then
 - The Vickrey imputation ($\pi_j = w(N) - w(N-j)$ for $j \neq 0$) is in the core and is the unique bidder optimal core imputation (Ausubel-Milgrom, 2002)
 - Truthful reporting is an *ex post* equilibrium of M
 - Shill bidding is not profitable at equilibrium in M
 - Excluding bidders from M reduces the seller's truthful equilibrium revenues



Other Cases are Hard

- Theorem (Ausubel-Milgrom, 2002). If $V \not\subset V_{\text{sub}}$ and $V_{\text{add}} \subset V$, then there exists a profile of valuations from V such that the Vickrey outcome is not a core outcome.
- Definitions
 - V is the set from which individual bidders' valuations of goods are drawn
 - V_{add} is the set of “additive” valuations
 - the value of a package is the sum of the item values
 - V_{sub} is the set of substitutes valuations
 - the implied demand function satisfies the substitutes condition
- The AWS auction example is NOT a case of substitutes.



Package Auction Design: Shills and Revenue Requirements

Yokoo, Sakurai & Matsubara (2000, 2001, 2004) worked on “false name bidders”—here called “shills.”



Shills and Merged Reports

- A set of shills S can instead report as a single entity by *merging* their reported values. The merged report is defined by:

$$u_S(z) = \max_{x \geq 0} \sum_{j \in S} u_j(x_j) \text{ subject to } \sum_{j \in S} x_j = z$$

- If a bidder makes a merged report, its Vickrey payoff is $w(N) - w(N-S)$.
- A core-selecting auction is a core-selecting direct mechanism for the quasi-linear payoff environment.



Shills, Payoffs and Core Selection

Theorem 1. Let M be direct mechanism.

- M always prescribes an efficient, individually rational allocation for every set of reports and never charges any set of bidders less than its merged Vickrey price
if and only if
- M is core-selecting.
- Discussion: so what?
 - Assumption: prices lower than Vickrey are unacceptable.
 - Question: when are such prices ruled out even when bidders can use shill strategies?



Proof

- Using skills S never reduces a bidder's price for the package it wins below the Vickrey price if and only if the reported profits always satisfy:

$$\sum_{j \in S} \pi_j \leq w(N) - w(N - S)$$

- Let $T = N - S$ be coalition including the seller. Then, the condition is restated as:

$$\sum_{j \in T} \pi_j \geq w(T)$$

for every coalition including the seller.

QED.



Two Core-Selecting Package Auctions

- Pay-as-bid (“menu”) package auction
 - Auction selects combination of bids that maximizes revenue. Bidders pay as bid.
 - Outcome is the seller-best core allocation for the reported preferences.
 - Bernheim & Whinston (1986), “Menu Auctions, Resource Allocation and Economic Influence.”
- Ascending proxy package auction
 - Ascending auction: final bids just sufficient to win.
 - Outcome is a core allocation with respect to reported values.
 - Vickrey outcome when goods are substitutes.
 - Ausubel & Milgrom (2002), “Ascending Auctions with Package Bidding.”



Stable Matching Analogy?

- Matching theory and applications emphasize stable allocations, which are NTU-core selecting mechanisms.
- Both the appeal of the class of core-selecting auctions and many of its properties are analogous to those of stable matching mechanisms.



Easy Connections

- Evidence about importance of stable/core-selecting mechanisms.
- Theorems that stable/core-selecting mechanisms are not generally strategy-proof.



Main Results

- “Truncation strategy” is a best reply in every stable matching mechanism/core-selecting auction.
 - Man-optimal, woman-optimal, student-optimal...
- Truncation strategy profiles that are equilibria for all stable matching mechanisms/core-selecting auction.
 - Resistance of these equilibria to shills.
- Selecting among stable matching mechanisms
 - Minimizing incentives to misreport... and
 - Eliminating incentives to exclude bidders... and
 - Eliminating value of shills



Importance of Stable Mechanisms



Stable Mechanisms in Practice

<u>Market</u>	<u>Stable</u>	<u>Still in use (halted unraveling)</u>
■ NRMP	yes	yes (new design in '98)
■ <i>Edinburgh</i> ('69)	yes	yes
■ <i>Cardiff</i>	yes	yes
■ <i>Birmingham</i>	no	no
■ <i>Edinburgh</i> ('67)	no	no
■ <i>Newcastle</i>	no	no
■ <i>Sheffield</i>	no	no
■ <i>Cambridge</i>	no	yes
■ <i>London Hospital</i>	no	yes
■ <i>Medical Specialties</i>	yes	yes (~30 markets, 1 failure)
■ <i>Canadian Lawyers</i>	yes	yes (Alberta, no BC, Ontario)
■ <i>Dental Residencies</i>	yes	yes (5) (no 2)
■ <i>Osteopaths</i> (< '94)	no	no
■ <i>Osteopaths</i> (\geq '94)	yes	yes
■ <i>Pharmacists</i>	yes	yes
■ <i>Reform rabbis</i>	yes (first used in '97-98)	yes
■ <i>Clinical psych</i>	yes (first used in '99)	yes
■ Lab experiments	yes	yes
■ (Kagel & Roth)	no	no



Why?

- Kagel/Roth: “There are good theoretical reasons and considerable empirical evidence to support the notion that, to successfully eliminate unraveling, a clearinghouse must either produce a stable matching, have power to compel compliance with all stages of the match, or have costs of going early that are high relative to the strategic advantages that going early provides.”
 - Idea concerns participation: If the mechanism is unstable, some parties will have an incentive to make deals early (or late).
 - Kagel & Roth (2000), “The Dynamics of Reorganization in Matching Markets: A Laboratory Experiment Motivated by a Natural Experiment.”
- The same argument would seem to apply to auctions that do not select core allocations.
- In both cases, the core-selecting mechanism could fail to select core allocations, due to false reporting of preferences.



Strategy-Proofness, Stability and the Core



...in Auctions & Matching

- Generally, there is no “stable” or “core-selecting” mechanism that is strategy-proof for all participants, even when goods are substitutes.
 - For the marriage problem, the mechanism that selects the “man-optimal match” is strategy-proof for men, but not for women.
 - For the auction problem, the Vickrey mechanism is strategy-proof for the bidders, but not for the seller.
 - ...one side of the market, but not the other...



Truncation Strategies as Best Replies

This theorem was independently reported by Robert Day and S. Raghavan, in “Fair Payments for Efficient Allocations in Public Sector Combinatorial Auctions,” forthcoming, Management Science and by Paul Milgrom, “Incentives in Core-Selecting Auctions.”



...in Matching & Auctions

- Matching (“Marriage Problem”)
 - Fix any reports for the women and for all the men besides man m .
 - Given man m 's actual preferences and others' reports, let w be the woman with whom man m is matched at his most preferred stable match.
 - Then, for every stable mechanism, man m has a best reply which is to report truthfully the rank ordering of the women, but to truncate: “any woman below w is unacceptable.”
 - Same best reply for every stable matching mechanism!
- Auctions
 - Fix the seller's reserve and the reports of all the bidders besides bidder m .
 - Given bidder m 's actual preferences and others' reports, let π be bidder m 's maximum core payoff, that is, m 's Vickrey payoff.
 - Then, for every core-selecting auction, bidder m has a best reply which is to report values for every package that are π less than the truthful values.
 - Same best reply for every core-selecting auction!



Proof of Theorem for Auctions

- Fix any core-selecting auction.
- Given j 's actual value and the reports of the other players, compute the coalitional value function w .
 - The value to the seller and the other bidders is at least $w(N-j)$; this does not depend on j 's report.
 - The “maximum total value” is $w(N)$, so j can earn no more than $\pi_j = w(N) - w(N-j)$ (which is j 's Vickrey payoff).
 - Assuming that $\pi_j > 0$, j can get into the winning set by reporting his actual values reduced by $\pi_j - \varepsilon$ for any $\varepsilon > 0$. Since j does not pay more than its value, j earns at least $\pi_j - \varepsilon$. (Then, do the Simon-Zame trick.)



An NTU Auction Theory?

- The results suggest that there should be an NTU model that nests both auctions and matching, which would be exciting!
 - Bidder budget constraints are very significant in many auctions, including FCC auctions.
 - These create NTU situations.
- ...but the general theory doesn't work...



Auctions with Budget Constraints

- Example with 3 bidders and 3 items.

Bidder	A	B	C	AB	BC	AC	ABC	Budget
1	10	10	0	17	10	10	17	≥ 17
2	0	0	0	14	19	19	19	14
3	0	0	9	0	9	9	9	≥ 9

- If 2 and 3 report values & budget truthfully, then
 - If 1 reduces its reported values by more than 3, then the allocation where 2 buys AB for 14 and 3 buys C for 9 is an NTU-core allocation.
 - If 1 reduces its reported values by 6, there is an NTU-core allocation at which 1 buys A at price 4 while 2 buys BC at price 14.
- Conclusion: In this example, 1's best truncation depends on which core-selecting auction is used.



Uniform Equilibrium Strategy Profiles



Truncation Equilibrium Profiles

- In both auction and matching theory, some theorems treat a subset of the players as strategic and exclude analysis of others' reporting incentives.
 - Auction theorems may exclude seller's incentive to set reserve.
 - Matching theorems may exclude incentives of men or of women.
 - Our analysis here focuses on any group of players S , fixing "truthful" reports for players not in S .
- Theorem. For any S -optimal (core) allocation x and any TU-core-selecting auction or stable man-woman matching mechanism, there is a Nash equilibrium at which players in S report that outcomes worse than x are unacceptable, but otherwise report relative ranks truthfully.
 - Examples: Bertrand equilibrium and Gale-Shapley matching



Shills Don't Break Nash Equilibrium

- Use two previous theorems.
- Theorem. For
 - every core-selecting auction, and
 - every bidder optimal allocation x and its corresponding profile of truncation strategies,
- no bidder has a profitable deviation either
 - in the original strategy set (Nash equilibrium) or
 - in the extended strategy set in which bidders can hire shills.



How Close to Straightforward?



Why do we care?

- If straightforward bidding does well, then
 - participation costs will be low.
 - learning to bid well may be quick.
 - bidders whose best reply problems are combinatorial may choose to bid straightforwardly, or nearly so.

- If bidders do bid (nearly) straightforwardly, then perhaps
 - outcome is (nearly) efficient
 - revenue is (nearly) competitive



Ex Post Equilibrium? (Rarely)

- Fix any coalition S and consider any auction mechanism that always picks an S -optimal (core) allocation.
 - Ex: man-optimal, woman-optimal, bidder-optimal
- Non-members of S are “non-strategic” and report truthfully.
- Theorem: In the auction game described above, for any value profile, if there is a unique S -optimal allocation, then truthful reporting by members of S is an ex post equilibrium.



Proof Sketch

- By our previous theorem, for any core-selecting mechanism, the best any player in S can do by deviating is to get its best core allocation, earning its Vickrey payoff.
- If truthful reporting selects that same core allocation, there is no *ex post* incentive to deviate.



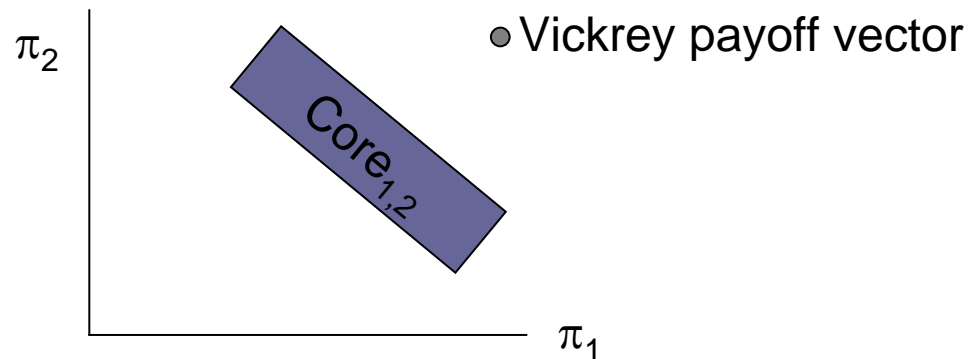
Minimal Incentives to Misreport

- Theorem: Let A^* be an S -optimal core-selecting auction. For any preference profile and any other core-selecting auction A , if there is some bidder in S who gains strictly more by misreporting in A^* than in A , then there is another bidder in S who gains strictly more by misreporting in A than in A^* .



Proof Sketch

- By a previous theorem, for any core-selecting mechanism, the most any player in S can earn by deviating in either A or A^* is its “Vickrey payoff.”
- So, the incentives to misreport vary inversely with the payoffs. Hence, if the mechanism selects an S -optimal allocation, then one cannot improve the incentive for some player in S without harming that of another player in S .



Revenue Monotonicity



Seller Pessimal Core Allocations

- “Seller pessimal” means revenue minimizing
- Consider the TU game (N, w) and the associated problem of minimization revenue over the core:

$$\min_{\pi} w(N) - \sum_{j \neq 0} \pi_j$$
$$\sum_{j \in S} \pi_j \geq w(S) \text{ for all } S \subseteq N, \text{ where}$$
$$w(S) = \begin{cases} 0 & \text{if } 0 \notin S \\ \max_{x \in X} \sum_{j \in S} b_j(x) & \text{if } 0 \in S \end{cases}$$

- By inspection,
 - w is non-decreasing in the bids b_j
 - The minimum is non-decreasing in w .



Seller Pessimal Auctions

Theorem.

- Every seller-pessimal allocation is a bidder-optimal allocation.
- An auction that always selects seller-pessimal core allocations
 - “Minimizes” bidder incentives to misreport
 - Eliminates seller incentives to exclude bidders.
- ...but there may be many seller-pessimal allocations.



Full Shill-Proofness

- An auction is fully shill-proof if, regardless of others' reports, a bidder cannot reduce its price for any collection of items by using shills instead of their merged report.
- Theorem. A seller-pessimal auction is fully shill-proof if, for any convex function f , it selects among the revenue-minimal core allocation by minimizing

$$\sum_{j \in N-0} f(p_j).$$

