Asymmetric Traveling Salesman Problem

We are given a directed graph $D = (V, A)$ equipped with non-symmetric distances $d : A \rightarrow \mathbb{R}_+$ which satisfy the triangle inequality $d(i, j) \leq d(i, k) + d(k, j)$, $\forall i, j, k \in V$. The goal is to find the Hamiltonian cycle $F$ which minimizes its length:

$$d(F) \triangleq \sum_{a \in F} d_a.$$

It is important to note that any Eulerian connected cycle is enough to obtain a Hamiltonian cycle, as one can shortcut it. An approximation of $O(\log n)$ was given by Frieze et. al., which was later improved to $O(\log n / \log \log n)$ by Asadpour et. al.

Consider the following relaxation for the problem:

$$\min \sum_{a \in A} c_a \cdot x_a$$

s.t. $\sum_{a \in \delta^+(v)} x_a = \sum_{a \in \delta^-(v)} x_a = 1 \quad \forall v \in V$

$\sum_{a \in \delta^+(S)} x_a \geq 1 \quad \forall S \subseteq V$

$x_a \geq 0 \quad \forall a \in A$

We consider the following natural algorithm which iteratively reduces the size of the graph repeatedly:

1. $F \leftarrow \emptyset$.
2. While $D$ is not a single vertex do:
   (a) Solve the LP on $D$ to get $x^*$.
   (b) $x^* = \sum_{C \text{ is a cycle cover of } D} \lambda_C \cdot \mathbf{1}_C$, where $\sum \lambda_C = 1$, $\lambda_C \geq 0$.
   (c) Sample a cycle cover $C^*$ with probability $\lambda_{C^*}$.
   (d) Find a representative vertex from each cycle in $C^*$ and delete all other vertices.

**Theorem 1.** The above algorithm gives an approximation of $O(\log n)$.

**Conjecture 1.** The above algorithm is in fact a $O(\log n / \log \log n)$-approximation.

**Conjecture 2.** The above algorithm is in fact a $O(1)$-approximation.