Nearly Optimal Partial Steiner Systems

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Abstract
A partial Steiner system \( S_p(t, k, n) \) is a collection of \( k \)-subsets (i.e. subsets of size \( k \)) of \( n \) element set such that every \( t \)-subset is contained in at most one \( k \)-subset. To avoid trivial cases, we assume \( 2 \leq t < k < n \). It is easy to see that the size of a partial Steiner system \( S_p(t, k, n) \) is at most \( \binom{n}{t}/\binom{k}{t} \). Confirming a conjecture of Erdős and Hanani [6], Rödl [21] proved that for fixed positive integers \( t \) and \( k \), there is a partial Steiner system \( S_p(t, k, n) \) of size at least \( (1-o(1)) \left( \frac{n^t}{k^t} \right) \), where \( o(1) \) goes to 0 as \( n \) goes to infinity. Grable [8] found an explicit bound of the \( o(1) \) term above and it was improved by Kostochka and Rödl [17] to \( n^{-1/(\binom{k}{t}+o(1)-1)} \). All proofs used Rödl’s nibble method. We also use the method to improve the bound to
\[
O\left( \left( \frac{\log n}{n} \right)^{\frac{k-t}{\binom{k}{t}-1}} \right)
\]
for \( 2 \leq t \leq k-2 \). (For \( t = k-1 \), slightly better bounds were already known. See [2].)
We believe that the nibble method gives no better bound up to a logarithmic factor.

1 Introduction
A hypergraph \((V, H)\) is a pair of a set \( V \) of vertices and a collection \( H \) of subsets of \( V \), which are called hyperedges or simply edges. It is \( k \)-uniform if each edge is of size \( k \). Among others, two special kinds of hypergraphs have attracted much attention. If some geometrical structures are required, then it becomes a subject of finite geometry. For example, one may consider projective and affine spaces. On the other hand, the theory of block design concerns certain numerical requirements. For example, a Steiner system \( S(t, k, n) \) is a \( k \)-uniform hypergraph on \( n \) vertices in which every set of \( t \) vertices, called a \( t \)-subset, is contained in exactly one edge. To avoid trivial cases, we will always assume \( 2 \leq t < k < n \). Since every edge, which is a \( k \)-subset, contains \( \binom{k}{t} \) \( t \)-subsets, a Steiner system \( S(t, k, n) \) has \( \binom{n}{t}/\binom{k}{t} \) edges. Similarly, a partial Steiner system \( S_p(t, k, n) \) is a \( k \)-uniform hypergraph on \( n \) vertices in which every \( t \)-subset is contained in at most one edge. Clearly, a partial Steiner system \( S_p(t, k, n) \) has at most \( \binom{n}{t}/\binom{k}{t} \) edges. This paper concerns partial Steiner systems of almost optimal sizes.
Confirming a conjecture of Erdős and Hanani [6], Rödl [21] proved

**Theorem 1.1** ([21]). For fixed positive integers $t$ and $k$, there is a partial Steiner system $S_p(t, k, n)$ of size

$$|S_p(t, k, n)| \geq (1 - o(1)) \left( \frac{n}{t} \right),$$

where $o(1)$ goes to 0 as $n$ goes to infinity.

The proof method was related to a random greedy construction, that is, randomly order all $k$-subsets, choose the first one, and then discard all $k$-subsets containing any $t$-subset of the chosen $k$-subset. Choose the next non-discarded one and do the same procedure. At the end of this construction, the collection of chosen $k$-subsets is clearly a partial Steiner system and expected to have almost optimal number of $k$-subsets. However, it was generally believed that estimating the size of it was very difficult though, using Rödl’s proof, the same lower bound as in Theorem 1.1 has been proved (see [22], [23]). Motivated by a seminal paper of Ajtai, Komlós and Szemerédi [1], Rödl considered an approximation of the random greedy construction, called a nibble method to prove Theorem 1.1. This method will be described later.

Theorem 1.1 has been generalized to certain hypergraphs by Frankl and Rödl [7] and later by Pippenger [19] (see also [20]). To state these results, some definitions are needed. A matching of a hypergraph is a collection of mutually disjoint edges. For example, a partial Steiner system $S_p(t, k, n)$ may be regarded a matching of the hypergraph $\mathcal{H}(k, n)$ in which a vertex is a $t$-subset and an edge is the collection of all $t$-subsets of a $k$-subset. An edge in $\mathcal{H}(k, n)$ may and will be regarded as a $k$-subset. The hypergraph $\mathcal{H}(k, n)$ is a $\binom{k}{t}$-uniform hypergraph on $\binom{n}{t}$ vertices. Generally, a matching of a $k$-uniform hypergraph on $n$ vertices is of size at most $n/k$. If a matching has exactly $n/k$ edges then it is called a perfect matching. A Steiner system $S(t, k, n)$ is a perfect matching of $\mathcal{H}(k, n)$. The degree $d(v)$ of a vertex $v$ denotes the number of edges containing it and the codegree $\text{codeg}(v, w)$ of a pair of distinct vertices $v, w$ denotes the number of edges containing both vertices. If all degrees are the same, say $D$, then the hypergraph is called $D$-regular.

**Theorem 1.2** ([19]) If $k$ is fixed and $D$ goes to infinity, any $k$-uniform $D$-regular hypergraph with $\max\{\text{codeg}(v, w)\} = o(D)$ has an almost perfect matching $M$, that is,

$$|M| \geq (1 - o(1))\frac{n}{k},$$

where $C = o(D)$ means that $C/D$ goes to 0 as $D$ goes to infinity.

Earlier, Frankl and Rödl proved the same result under a stronger condition $O(D/(\log n)^3)$ for the maximum codegree. Far reaching generalization to edge colorings and list-colorings can be found in Pippenger and Spencer [20], and Kahn [11, 12].

Efforts to make the $o(1)$ term more explicit and/or smaller have been also made. Let $\mathcal{H}$ be a $k$-uniform $D$-regular hypergraph on $n$ vertices with the maximum codegree $C$, where, as before, $k$ is fixed and $D$ goes infinity. Grable [8] obtained an explicit bound in terms of $C\log n/D$.

**Theorem 1.3** [8] If $C\log n = o(D)$, then there is a matching $M$ of $\mathcal{H}$ such that

$$|M| \geq \left(1 - O\left(\left(\frac{C\log n}{D}\right)^{1/(2k-1+o(1))}\right)\right)\frac{n}{k}.$$
As a corollary, there is a $S_p(t, k, n)$ of size at least

$$\left(1 - \frac{1}{n^{1/(2(t_k)^{-1+o(1)})}}\right) \binom{n}{t} / \binom{k}{t}.$$ 

When $C = 1$, i.e., $H$ is simple, Alon, Kim and Spencer improved Grable’s bound.

**Theorem 1.4** ([2]) If $C = 1$, then there is a matching $M$ of $H$ such that

$$|M| \geq (1 - O(1/D^{1/(k-1)})) \frac{n}{k} \quad \text{if } k > 3$$

$$|M| \geq (1 - O(\log^{3/2} D/D^{1/2})) \frac{n}{k} \quad \text{if } k = 3.$$ 

Thus there is a $S_p(k - 1, k, n)$ of size at least

$$\frac{1 - O(1/n^{1/(k-1)})}{k} \binom{n}{k-1} \quad \text{if } k > 3$$

$$\frac{1 - O(\log^{3/2} n/n^{1/2})}{3} \binom{n}{2} \quad \text{if } k = 3,$$

which, in particular, improves Brouwer’s [5] bound of $(1 - O(n^{-1/3})) (\binom{n}{3})/3$ for Steiner triple systems $S_p(2, 3, n)$. The theorem has been further generalized by Kostochka and Rödl to the case that $C$ is significantly smaller than $D$.

**Theorem 1.5** ([17]) If $C \leq D^{1-\delta}$ for some fixed $\delta > 0$, then there is a matching of size at least

$$(1 - O((C/D)^{1/(k-1-o(1))})) n/k.$$ 

This theorem implies that there is a $S_p(t, k, n)$ of size at least

$$\left(1 - \left(\frac{1}{n}\right)^{1/(\binom{k}{t}^{-1+o(1)})}\right) \frac{n}{t} / \binom{k}{t}.$$ 

In this paper, we focus on only partial Steiner systems and find better bounds.

**Theorem 1.6** Let $t, k$ be fixed with $2 \leq t \leq k - 2$. Then there is a partial Steiner system $S_p(t, k, n)$ of size at least

$$\left(1 - O\left(\frac{\log n}{n} - (k-t)/(\binom{k}{t}-1)\right)\right) \frac{n}{t} / \binom{k}{t}.$$ 

**Remark.** Notice that $t = k - 1$ cases were covered in Theorem 1.4.

**Nibble Methods**

As mentioned earlier, to construct a partial Steiner system one may randomly order all $k$-subsets and choose one at a time according to the order only if it does not contain any $t$-subset of previously chosen $k$-subsets. It would be discarded otherwise. Since this construction is hard to analyze,
instead of choosing one \( k \)-subset at a time, we may choose each \( k \)-subset with a certain probability independently of others, so that a bunch of \( k \)-subsets will be chosen at a time. There may be some conflicts among chosen \( k \)-subsets, namely two such \( k \)-subsets may share more than \( t-1 \) vertices. In this case, the two would be returned. If a \( k \)-subset is chosen and not returned, it is called selected. Discard now all remaining \( k \)-subsets that contain any \( t \)-subset of a selected \( k \)-subset. Since many \( k \)-subsets are to be selected at a time, large deviation type inequalities may be used and hence we may quite well understand the structure of all surviving, i.e., neither selected nor discarded, \( k \)-subsets. This was not possible when only one \( k \)-subset was chosen at a time. Azuma-Heoffding [4, 9] type martingale inequalities play nice roles here. This procedure is called a nibble or a bite depending on its size. We will maintain certain conditions to repeatedly apply the procedure. For this purpose, a set of waste vertices is introduced to stabilize degrees of surviving vertices in the induced subhypergraph on the set of surviving vertices.

After Rödl proved Theorem 1.1, more sophisticated constructions and martingale inequalities have been developed to settle intriguing problems including hypergraph edge coloring [20] and list-coloring [12], sparse graph coloring [13, 10], Ramsey number \( R(3,t) \) [14], total coloring [18], and complete arc of a projective plane [15]. Our method is similar to one developed in [2].

In the next section, we use the nibble method to randomly construct a partial Steiner system and present a main lemma, and then a proof of Theorem 1.6 modulo the main lemma. Section 3 is for martingale inequalities. The main lemma is then proved in the last section.

2 Random Construction and Main Lemma

The central idea of the construction is to repeatedly take a “random bite” out of the hypergraph \( \mathcal{H}(k,n) \). Recall that vertices of \( \mathcal{H}(k,n) \) are \( t \)-subsets and edges are regarded as \( k \)-subsets. Two edges \( E, E' \) are \( t \)-disjoint if \( |E \cap E'| < t \). Similarly, an edge is \( t \)-isolated, or simply isolated, in a collection of edges if it is \( t \)-disjoint with any other edge in the collection.

For a given subhypergraph \( \mathcal{H} = (V,H) \) of \( \mathcal{H}(k,n) \) with certain properties which will be specified later, we will take a pair \((W,X)\) of a set of vertices and a set of edges. Let \( M \) be the set of all \( t \)-isolated edges in \( X \), and

\[
V^* = V - \bigcup_{E \in M} \binom{E}{t} - W, \\
H = H|_{V^*} := \{E \in H : E \subseteq V^*\},
\]

where \( \binom{E}{t} \) is the set of all \( t \)-subsets of \( E \). It will be shown that \( \mathcal{H}' \) satisfies similar properties so that we may iteratively construct \( M^* \) from \( \mathcal{H}' \). Starting with \( \mathcal{H}_0 = \mathcal{H}(n,k) \), this construction will continue until only few vertices remain uncovered. Since the vertices in \( W \) may be covered too, we also need to show that the number of those vertices is small. The pair \((W,X)\) is to be randomly constructed.

For \( t \leq s \leq k \) and an \( s \)-subset \( S \subseteq V \), define degree \( \deg_s(S) \) of \( S \) in \( \mathcal{H} \) to be the number of edges in \( \mathcal{H} \) containing \( S \). Regardless whether \( S \subseteq V^* \) or not, \( \deg_s(S) \) denotes the number of edges \( E \) in \( \mathcal{H} \) containing \( S \) and \( \binom{E}{t} \setminus \binom{S}{t} \subseteq V^* \). Suppose \( \mathcal{H} \) satisfies

\[
D_t - \Delta \leq \deg_t(x) \leq D_t
\]

for all \( t \)-subsets and \( x \in V \). The parameters \( D_t, \Delta \) will be specified later. A random set \( X \) of edges is defined as follows. For any edge \( E \),

\[
\Pr[E \in X] = 1/D_t,
\]
independently of any other events. Let $M$ be the set of $t$-isolated edges in $X$. We will say that $x$ is covered by $M$ and write $x < M$ if $x \in E$ for some $E \in M$. (Since we regard an edge $E$ as a $k$-subset and a vertex is a $t$-subset, we used $x \in E$.) Define

$$p(x) = \Pr[x < M] \quad \text{and} \quad p^* = \max_{x \in V} p(x).$$

The set $W$ of waste vertices is to be defined independently of $X$ so that

$$\Pr[x < M \ or \ x \in W] = p^*,$$

or equivalently,

$$p(x) + \Pr[x \in W] - p(x) \Pr[x \in W] = p^*,$$

independently of one another. Since $p(x)$ is determined by $\mathcal{H}$ not by the actual $X$, $W$ may be defined independently of $X$.

This enable us to assure

$$\Pr[x \in V^*] = 1 - p^*$$

for all $x \in V$. We expect that $p^*$ is approximately $e^{-\binom{t}{3}}$. Let $c(x) = \Pr[x \in W]$ (c for compensatory).

**Properties** We expect that the hypergraph $\mathcal{H}^*$ has almost the same properties as the hypergraph induced on a random vertex set to which each vertex independently belongs with probability $1 - p^*$. In particular, for all $t \leq s \leq k$,

$$\deg^*_s(S) \approx (1 - p^*)\left(\binom{t}{3} - \binom{s}{3}\right) \deg_s(S).$$

To be more precise, suppose that for all $t \leq s \leq k$, $s$-subsets $S$, and $x \in V$,

$$|V| \leq N, \quad \deg_s(S) \leq D_s \quad \text{and} \quad \deg_t(x) \geq D_t - \Delta$$

and

$$N \geq n^{t-1} \quad \text{and} \quad D_s \geq (K \log n)^{k-s},$$

where $\Delta := K(D_{t+1}D_t \log n)^{1/2}$ and a (large) positive constant $K$ depending only on $t$ and $k$. Then for $c = K^{1/3}$

$$|W| \leq cN(\log n D_{t+1}/D_t)^{1/2},$$

$$|V^*| \leq N^* := \left(1 - e^{-\binom{t}{3}}\right)\left(1 + c(D_{t+1} \log n/D_t)^{1/2}\right)(1 + (\log n)^{-2})N,$$

$$\deg^*_s(S) \leq D^*_s := \left(1 - e^{-\binom{t}{3}}\right(\binom{k}{3} - \binom{s}{3}\right)\left(1 + c(D_{s+1} \log n/D_s)^{1/2}\right)D_s,$$

$$\deg^*_t(x) \geq D^*_t - \Delta^* ,$$

where $\Delta^* := K(D^*_{t+1}D^*_t \log n)^{1/2}$. The lower bound of $\deg_t(x)$ is needed to bound the size $|W|$ of wasted vertices and (3) is needed to keep $(D_{s+1} \log n/D_s)^{1/2}$, which may be called deviation errors, small enough.

**Lemma 2.1 (Main Lemma)** Suppose a subhypergraph $\mathcal{H}$ of $\mathcal{H}_0$ satisfies (2) and (3), Then, with positive probability the random pair $W, X$ described above simultaneously satisfies all of (4). In particular, such a pair exists.
Proof of Theorem 1.6 (modulo Main Lemma). We repeatedly apply Main Lemma. To maintain (3), we need to set $D_s = (K \log n)^{k-s}$ as soon as $D^*_s \leq (K \log n)^{k-s}$. We first show that the deviation error terms $1 + c(\log n (D_{s+1}^{(i)} / D_s^{(i)}))^{1/2}$ are essentially negligible so that the construction continues until $D_t = (K \log n)^{k-t}$. For $t \leq s \leq k-1$ let $D_s^{(0)} = \binom{n-s}{k-s}$ and

$$D_s^{(i+1)} = \max \left\{ (K \log n)^{k-s}, \left(1 - e^{-\left(\frac{k}{t}\right)}\right)^{(i)} \left(1 + c(\log n (D_{s+1}^{(i)} / D_s^{(i)}))^{1/2}\right) D_s^{(i)} \right\},$$

where $D_k^{(i)} := 1$. Similarly, define $N^{(0)} = \binom{n}{t}$, and

$$N^{(i+1)} = \left(1 - e^{-\left(\frac{k}{t}\right)}\right) \left(1 + c(D_{t+1} \log n / D_t)^{1/2}\right) (1 + (\log n)^{-2}) N^{(i)}.$$

Denote $i_s$ to be the largest $i$ such that $D_s^{(i)} > (K \log n)^{k-s}$, $t \leq s \leq k-1$. The construction will stop after $i_t$ steps. Since $D_k^{(i)} = 1$ and $D_{k-1}^{(i)}$ geometrically decreases up to $i_{k-1}$, it is easy to see that

$$\prod_{i=1}^{i_{k-1}} \left(1 + c(\log n (D_k^{(i)}/D_{k-1}^{(i)}))^{1/2}\right) = O(1)$$

and

$$i_{k-1} = \frac{-\log n}{\left(\binom{k}{t} - \binom{k-2}{t}\right)} \log \left(1 - e^{-\left(\frac{k}{t}\right)}\right) + O(\log \log n).$$

Notice also that

$$\log D_{k-2}^{(i_{k-1})} \geq \left(2 - \frac{k}{t} - \frac{2}{t}\right) \log n + O(\log \log n) > \delta \log n,$$

for any fixed $\delta > 0$ satisfying

$$k - s + 1 - \frac{(k-s)(\binom{k}{t} - \binom{s-1}{t})}{\binom{k}{t} - \binom{s}{t}} > \delta \quad \forall s = k-1, \ldots, t+1.$$

Since $D_{k-1}^{(i)}/D_{k-2}^{(i)}$ increases, we have

$$\log(D_{k-1}^{(i)}/D_{k-2}^{(i)}) \leq -\log D_{k-2}^{(i_{k-1})} + O(\log \log n) < -\delta \log n$$

for $i \leq i_{k-1}$. Applying the same argument repeatedly using

$$D_{s+1}^{(i)}/D_s^{(i)} \leq n^{-\delta} \quad \forall i \leq i_{s+1},$$

we know that $D_s^{(i)}$ decreases geometrically up to $i_s$, and

$$\prod_{i=1}^{i_s} \left(1 + c(\log n (D_{s+1}^{(i)}/D_s^{(i)}))^{1/2}\right) = O(1),$$

(5)

and

$$i_s = \frac{- (k-s) \log n}{\left(\binom{k}{t} - \binom{s}{t}\right) \log \left(1 - e^{-\left(\frac{k}{t}\right)}\right)} + O(\log \log n).$$
In particular,
\[
(1 - e^{-\left(\frac{k}{t}\right)})^{i_t(i_t^{(k)}-1)} = \Theta\left(\left(\frac{\log n}{n}\right)^{k-t}\right),
\]
which implies that
\[
N^{(i_t)} = \Theta\left(\left(\frac{\log n}{n}\right)^{k+1-t} \binom{n}{t}\right).
\]
Therefore,
\[
N^{(i)} \geq N^{(i_t)} \geq n^{t-1}
\]
(see (3)) and the number of surviving \(t\)-subsets after step \(i_t\) is small enough as desired.

We now estimate the number of wasted vertices, or \(t\)-subsets. For \(i \leq i_{t+1}\), (5) implies that
\[
N^{(i)}(D^{(i)}_{t+1}/D^{(i)}_{t})^{1/2}
\]
geometrically increases (not strictly if \(t=2\)) with the ratio
\[
\left(1 - e^{-\left(\frac{k}{t}\right)}\right)^{1+((\binom{t}{i})-(\binom{t+1}{i})-(\binom{k}{i})+1)/2} = \left(1 - e^{-\left(\frac{k}{t}\right)}\right)^{-\left(t/2-1\right)}
\]
extcept the negligible factors 1 + \(c(\log n) D^{(i)}_{t+1}/D^{(i)}_{t}\). For \(i \geq i_{t+1}\), \(N^{(i)}(D^{(i)}_{t+1}/D^{(i)}_{t})^{1/2}\) further (strictly) increases with the ratio
\[
\left(1 - e^{-\left(\frac{k}{t}\right)}\right)^{-\left(\binom{t}{i}\right)/2+3/2}
\]
extcept the negligible factors for \(2 \leq t \leq k - 2\). Since \(i_t - i_{t+1} = \Omega(\log n)\),
\[
\sum_{i=1}^{i_t} N^{(i)}(D^{(i)}_{t+1}/D^{(i)}_{t})^{1/2} = O(N^{(i_t)}(D^{(i_t)}_{t+1}/D^{(i_t)}_{t})^{1/2}) = O(N^{(i_t)}).
\]

\[
\square
\]

3 Martingale Inequalities

In recent years, martingale inequalities have become one of the most powerful tools for the probabilistic method. A general discussion is given in [3] and more refined versions can be found in [12, 13, 14]. All of these references deal with static cases in which the order of the martingale exposures are fixed \textit{a priori} (see e.g. [13]) or irrelevant. In [2], a martingale inequality developed using an order of martingale exposures depending the values of already exposed random variables. This kind of inequality may be called a dynamic martingale inequality. This section develops a more general dynamic martingale inequality which suits our cases.

We assume our underlying probability space is generated by a finite set of mutually independent random variables \(\{\tau_i\}_{i \in [r]}\), where, as usual, \([r] = \{1, 2, \ldots, r\}\). Though we deal with general random variables, the readers could assume for simplicity that the random variables are Yes/No choices, which is enough for our cases. (In our cases, the choices are of the form \(E \in X \text{ and } w \in W\).)
Suppose \(Y\) is a random variable on the space, For example, \(Y\) may be \(|V^*|, |W|\), or \(\text{deg}_a(S)\).

We consider a solitaire game in which Paul finds the value of \(Y\) by making queries of an always truthful oracle Carole. A query is always a choice \(i \in [r]\) which can depend on Carol’s previous answers. Then Carole gives the value of random variable \(\tau_i\). A strategy for Paul can naturally be represented in a decision tree form. A path from the root in the decision tree is called a \textit{sequence of queries}. Since one need not to ask the same \(i\) query twice, we assume that no \(i\) appears twice in a sequence of queries. Without loss of generality, we may also assume that all leaves of the
decision tree have the same distance from the root, for we might attach appropriate extra vertices to the tree otherwise. The length of a strategy is the depth of the corresponding decision tree. A full sequence of queries of a strategy is a sequence of queries of which length is the depth of the strategy. More precisely, a strategy is a collection random variables \( \{q_j\} \), \( q_j = q_j(\tau_{q_1}, \ldots, \tau_{q_{j-1}}) \in [r] \) with \( q_j, q_{j'} \), \( j \neq j' \), always distinct.

Define
\[
Z_j = Z_j(\tau_{q_1}, \ldots, \tau_{q_j}) := E[Y|\tau_{q_1}, \ldots, \tau_{q_j}] - E[Y|\tau_{q_1}, \ldots, \tau_{q_{j-1}}],
\]
and denote the variance \( \text{Var} j = \text{Var} j(\tau_{q_1}, ..., \tau_{q_{j-1}}) \) of the choice \( \tau_{q_j} \) given \( \tau_{q_1}, ..., \tau_{q_{j-1}} \), by
\[
\text{Var} j := E[(Z_j)^2|\tau_{q_1}, ..., \tau_{q_{j-1}}]
\]

Roughly speaking, Paul’s goal is to find a strategy which makes the sums \( \sum_{i=1}^r \text{Var} i \) small and then obtain a tight upper bound of \( E[e^{\lambda Y}] \) for suitable \( \lambda > 0 \). For a given strategy of length \( r \), variance bounds of the strategy are constants \( C_1, C_2 \), and functions \( C_j = C_j(\tau_{q_1}, ..., \tau_{q_{j-2}}) \), \( j = 2, \ldots, r \), such that for \( j = 1, \ldots, r \),
\[
\text{Var} j + C_{j+1} \leq C_j, \tag{6}
\]
where \( C_{r+1} = 0 \). These inequalities will enable us to estimate \( E[e^{\lambda Y}] \). The total variance \( \sigma^2 \) of a strategy with variance bounds \( \{C_i\} \) is \( C_1 \). Paul’s goal is to find a strategy and good variance bounds of it, in the sense that \( \sigma^2 \) is small. Note that we need to check inequalities (6) only for all sequences of queries of a strategy. This is a crucial savings, as we may impose some nice conditions on the sequences which can reduce the total variance.

**Lemma 3.1** Suppose Paul has a strategy finding \( Y \) with variance bounds \( \{C_i\} \). If \( \varepsilon, \lambda > 0 \) satisfy \( \lambda \max_j \sup_i |Z_i| \leq \log(1 + \varepsilon) \) for all \( i = 1, \ldots, r \), then
\[
E[e^{\lambda(Y-E[Y])}] \leq e^{\lambda^2(1+\varepsilon)\sigma/2}
\]
and similarly,
\[
E[e^{\lambda(E[Y]-Y)}] \leq e^{\lambda^2(1+\varepsilon)\sigma/2}.
\]

**Proof.** Since \( Y - E[Y] = \sum_{j=1}^r Z_i \). We claim that, for all \( s = 1, \ldots, r \),
\[
E[e^{\lambda \sum_{j=1}^r Z_j}] \leq E[e^{\lambda^2(1+\varepsilon)C_{s+1}/2} e^{\lambda \sum_{j=1}^{s-1} Z_j(q)}].
\]
The case \( s = r \) is trivial. Suppose we have the inequality for \( s \leq r \). Then, since \( C_{s+1} \) and \( \sum_{j=1}^{s-1} Z_j \) depends only upon \( \tau_{q_1}, \ldots, \tau_{q_{s-1}} \),
\[
E[e^{\lambda^2(1+\varepsilon)C_{s+1}/2} e^{\lambda \sum_{j=1}^{s-1} Z_j}] = E \left[ E[e^{\lambda^2(1+\varepsilon)C_{s+1}/2} e^{\lambda \sum_{j=1}^{s-1} Z_j}|\tau_{q_1}, \ldots, \tau_{q_{s-1}}] \right]
\]
\[
= E \left[ e^{\lambda^2(1+\varepsilon)C_{s+1}/2} e^{\lambda \sum_{j=1}^{s-1} Z_j} E[e^{\lambda Z_{s+1}|\tau_{q_1}, \ldots, \tau_{q_{s-1}}}] \right].
\]

Since \( E[Z_i|\tau_1, \ldots, \tau_{s-1}] = 0 \), Taylor expansion implies that there is \( \lambda^* \) between 0 and \( \lambda \) satisfying
\[
E[e^{\lambda Z_{s+1}|\tau_{q_1}, \ldots, \tau_{q_{s-1}}}] = 1 + \frac{\lambda^2}{2} E[Z_{s+1}^2 e^{\lambda^* Z_{s+1}|\tau_{q_1}, \ldots, \tau_{q_{s-1}}}] \\
\leq 1 + \frac{\lambda^2}{2} E[Z_s^2 e^{\lambda \max\{Z_s\}|\tau_{q_1}, \ldots, \tau_{q_{s-1}}}] \\
\leq 1 + \frac{\lambda^2(1+\varepsilon)}{2} \text{Var}_s \leq e^{\lambda^2(1+\varepsilon)\text{Var}_s/2}.
\]
In conclusion,
\[
\mathbb{E}[e^{\lambda \sum_{j=1}^{r} Z_j}] \leq \mathbb{E}[e^{\lambda^2(1+\varepsilon)C_{r+1}/2}e^{\lambda \sum_{j=1}^{r} Z_j}]
\leq \mathbb{E} \left[ (e^{\lambda^2(1+\varepsilon)C_{r+1}/2}e^{\lambda (\sum_{j=1}^{r} Z_j)} (e^{\lambda^2(1+\varepsilon)\text{Var}_r/2})) \right]
\leq \mathbb{E} \left[ e^{\lambda^2(1+\varepsilon)C_{r}/2}e^{\lambda \sum_{j=1}^{r} Z_j} \right].
\]

Lemma 3.2 (Dynamic Martingale Inequality) Suppose Paul has a strategy finding \(Y\) with variance bounds \(\{C_i\}\), and \(\varepsilon, \alpha > 0\) satisfy \(\alpha \max_j \sup |Z_j| \leq \sigma(1+\varepsilon)\log(1+\varepsilon)\) for all \(i\). Then

\[
\Pr[|Y - \mathbb{E}[Y]| > \alpha \sigma] \leq 2 \exp \left( -\frac{\alpha^2}{2(1+\varepsilon)} \right).
\]

Proof. Lemma 3.1 and Markov’s inequality yield

\[
\Pr[Y - \mathbb{E}[Y] > \alpha \sigma] = \mathbb{E}[e^{\lambda(Y - \mathbb{E}[Y])}] \leq e^{-\lambda \alpha \sigma + \lambda^2(1+\varepsilon)\sigma^2/2}.
\]

Take \(\lambda = \frac{\alpha}{\sigma(1+\varepsilon)}\) to have

\[
\Pr[Y - \mathbb{E}[Y] > \alpha \sigma] \leq \exp \left( -\frac{\lambda^2}{2(1+\varepsilon)} \right).
\]

Similarly,

\[
\Pr[\mathbb{E}[Y] - Y > \alpha \sigma] \leq \exp \left( -\frac{\lambda^2}{2(1+\varepsilon)} \right).
\]

We write \(\tau \equiv j \tau’\) when \(\tau = (\tau_j)\) and \(\tau’ = (\tau’_j)\) are the same but possibly the coordinate \(j\). Suppose, for each \(j \in [r]\), there is \(U_j = U_j(\tau_1, \tau_2, ..., \tau_{j-1}, \tau_{j+1}, ..., \tau_r)\) such that

\[
|Y(\tau) - Y(\tau’)| \leq U_j(\tau_1, \tau_2, ..., \tau_{j-1}, \tau_{j+1}, ..., \tau_r), \quad \text{for all } \tau \equiv j \tau’.
\]

Then it is easy to see that

\[
\sup_{\tau_{q_j}} Z_j(\tau_{q_1}, ..., \tau_{q_j}) - \inf_{\tau_{q_j}} Z_j(\tau_{q_1}, ..., \tau_{q_j}) \leq \mathbb{E}[U_j | \tau_{q_1}, ..., \tau_{q_{j-1}}],
\]

and

\[
\text{Var}_j(\tau_{q_1}, ..., \tau_{q_{j-1}}) \leq p_{q_j} (1 - p_{q_j}) \left( \mathbb{E}[U_j | \tau_{q_1}, ..., \tau_{q_{j-1}}] \right)^2,
\]

where \(p_{q_j} = \Pr[\tau_{q_j} = 1]\). Also, it is easy to see \(\sup |Z_j| \leq \sup |U_j|\). The next lemma easily follows from Lemma 3.2.

Lemma 3.3 Suppose (7). If there are a strategy with length \(r\), and \(\{C_j = C_j(\tau_{q_1}, ..., \tau_{q_{j-1}})\}^{r+1}_{j=1}\) with \(\sigma^2 = C_1\), \(C_{r+1} = 0\) satisfies

\[
p_{q_j} (1 - p_{q_j}) \left( \mathbb{E}[U_j | \tau_{q_1}, ..., \tau_{q_{j-1}}] \right)^2 + C_{j+1}(\tau_{q_1}, ..., \tau_{q_{j-1}}) \leq C_j(\tau_{q_1}, ..., \tau_{q_{j-2}}),
\]

for all \(j = 1, ..., r\), then \(\{C_i\}\) are variance bounds of the strategy and

\[
\Pr[|Y - \mathbb{E}[Y]| > \alpha \sigma] \leq 2 \exp \left( -\frac{\alpha^2}{2(1+\varepsilon)} \right),
\]

for all \(\varepsilon, \alpha > 0\) satisfying \(\alpha \max_j \sup |U_j| \leq \sigma(1+\varepsilon)\log(1+\varepsilon)\).

\[\square\]
4 Proof of Main Lemma

We first show that

\[ p(x) = e^{-\binom{k}{t}} \left( 1 + O \left( (D_{t+1} \log n / D_t)^{1/2} \right) \right). \]  \hspace{1cm} (10)

**Proof.** Since \( M \) consists of all isolated edges in \( X \),

\[ p(x) = \Pr[x \prec M] = \sum_{E \ni x} \Pr[E \in M] = \sum_{E \ni x} \frac{1}{D_t} (1 - \frac{1}{D_t})^{\varphi(E)}, \]

where \( \varphi(E) \) is the number of \( E' \in \mathcal{H} \setminus \{ E \} \) containing a common vertex with \( E \). Notice that

\[ \varphi(E) \leq \sum_{y \in E} (\deg_t(y) - 1) \leq \binom{k}{t} D_t \]

and, by (2),

\[ \varphi(E) \geq \sum_{y \in E} (\deg_t(y) - 1) - \sum_{y,z \in E \ y \neq z \ codeg(y,z)} \]
\[ \geq \binom{k}{t} (D_t - \Delta - 1 - O(D_{t+1})) \]
\[ = \binom{k}{t} D_t - O(\Delta). \]

Therefore,

\[ p(x) = (D_t - O(\Delta)) \frac{1}{D_t} (1 - \frac{1}{D_t})^{\binom{k}{t} D_t - O(\Delta)} = e^{-\binom{k}{t}} (1 + O(\Delta / D_t)) = e^{-\binom{k}{t}} \left( 1 + O((D_{t+1} \log n / D_t)^{1/2}) \right). \]

\[ \square \]

**Wasted vertices.** Since (10) gives

\[ p^* = e^{-\binom{k}{t}} \left( 1 + O((D_{t+1} \log n / D_t)^{1/2}) \right), \] \hspace{1cm} (11)

we have

\[ c(x) = O((D_{t+1} \log n / D_t)^{1/2}), \]

and

\[ \mathbb{E}[|W|] = \sum_{x \in V} c(x) = O(N(D_{t+1} \log n / D_t)^{1/2}). \]

Moreover, since \( |W| \) is the sum of mutually independent indicator random variables, we easily have

\[ \Pr[|W| \geq c_1 N(D_{t+1} \log n / D_t)^{1/2}] \leq e^{-N(D_{t+1} \log n / D_t)^{1/2}} = o(1), \]

for an appropriate large constant \( c_1 \).
For $|V^*|$ and $deg^*_S(S)$, we first derive inequalities like (7).

**Limited Effect.** Consider big collection $X$ of edges and small collection $X'$ which are identical except for one $E \in H$, chosen in $X$ but not chosen in $X'$. If $E$ is isolated in $X$ then it is in $M$ but not in $M'$. On the other hand, if $E$ is not isolated in $X$ and $M \neq M'$ then there is an edge $E' \in X'$ with $|E' \cap E| \geq t$ isolated in $X'$ but not in $X$. There are at most $\binom{k}{t}$ such $E'$ since two $E'$ containing a common vertex (i.e. $t$-subset) cannot both be isolated and $E$ contains $\binom{k}{t}$ vertices. Hence $M$ and $M'$ differs at most $\binom{k}{t}$ edges so that $|V^*|$ differs by at most $\binom{k}{2}$. To deal with $deg^*_S(S)$, we first denote the codegree of $x \in V \setminus S$ and $S$ by

$$codeg_x(S,x) := |\{F \in H : S \subseteq F, x \in F\}| .$$

For notational convenience, we define $codeg_x(S,x) = 0$ if $x \in S$. Then difference $deg^*_S(S)$ between $X$ and $X'$ is at most upper bounded by the primary effect, the effect when $E$ is isolated in $X$,

$$\sum_{x \in E} codeg_x(S,x)$$

or the secondary effect, the effect when $E$ is not isolated in $X$,

$$\sum_{E' \setminus \{E' \cap E\} \geq t} \sum_{x \in E'} codeg_x(S,x) .$$

Suppose now $W$ and $W'$ differ only in that $x \in W$ but $x \notin W'$ for a fixed vertex $x$. This affects $V^*$ by at most one vertex, namely $x$ itself. The effect of $x$ on $deg^*_S(S)$ is at most $codeg_x(S,x)$.

To represent the above result using the notations developed in the previous section, define

$$\tau_E = \begin{cases} 1 & \text{if } E \in X \\ 0 & \text{if } E \notin X, \end{cases}$$

for $E \in \mathcal{H}$, and

$$\tau_x = \begin{cases} 1 & \text{if } x \in W \\ 0 & \text{if } x \notin W, \end{cases}$$

for $x \in V$. All random variables are to be functions of $\tau = ((\tau_E),(\tau_x))$. Suppose $\tau$ and $\tau'$ are identical except for one edge $E \in H$, that is, $\tau \equiv_E \tau'$, say $\tau$ is the larger. Then we know

$$|V^*(\tau) - V^*(\tau')| = \binom{k}{t} = O(1) .$$

Thus (8) implies that the variances for $Y = V^*$ satisfy

$$Var_E = O(1/D_t) , \quad Var_x \leq c(x)$$

independently of queries and $E$. For $Y := deg^*_S(S)$, we had

$$|Y(\tau) - Y(\tau')| \leq \sum_{x \in E} codeg_x(S,x) + \sum_{E' \setminus \{E' \cap E\} \geq t} \sum_{x \in E'} codeg_x(S,x) ,$$

where $X'$ is with respect to $\tau'$. Set

$$U_E^{(p)} := \sum_{x \in E} codeg_x(S,x)$$

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and
\[ U_E^{(s)}(\tau) := \sum_{E' : |E' \cap E| \geq t} \sum_{x \in E'} \text{codeg}_s(S, x). \]

Clearly, \( U^{(s)} \) does not depend on \( \tau \). As before, \( U_E^{(p)} \) is called primary effect and \( U_E^{(s)} \) secondary effect of \( E \). Denote
\[ U_E := U_E^{(p)} \mathbb{I}(E : \text{isolated in } X) + U_E^{(s)}, \]
where
\[ \mathbb{I}(E : \text{isolated in } X) = \begin{cases} 1 & \text{if } |E \cap E'| < t \text{ for all } E' \in X' \\ 0 & \text{otherwise.} \end{cases} \]

Then, for all \( \tau \equiv E \tau' \),
\[ |Y(\tau) - Y(\tau')| \leq U_E \] (13)
(cf. (7)). Similarly, for \( x \in V \) and \( \tau \equiv x \tau' \),
\[ |Y(\tau) - Y(\tau')| \leq U_x := \text{codeg}_s(S, x). \] (14)

Upper bounds of \( U_E^{(p)} \) and \( U_E^{(s)} \) can be easily obtained as follows. Notice that \( \text{codeg}_s(S, x) = 0 \) if \( x \in S \) and for \( x \not\in S \), (2) gives \( \text{codeg}_s(S, x) \leq \deg|_{S \cup x}(S \cup x) \leq D_{s+1} \). Thus
\[ U_x \leq D_{s+1}, \] (15)
and
\[ U_E^{(p)} = O(D_{s+1}), \] (16)
uniformly in \( E \). Moreover, as we noted before,
\[ \sum_{E' : |E' \cap E| \geq t} \sum_{x \in E'} 1 \leq \left(\frac{k}{t}\right)^2 \]
and
\[ U_E^{(s)} = O(D_{s+1}), \] (17)
uniformly in \( E \). In what follows, all \( O(\cdot) \) statements will hold uniformly and we do not explicitly mention it.

**Vertices.** It is already shown that
\[ \mathbb{E}[|V^*|] = (1 - p^*)|V|. \]

Our probability space is determined by at most \( ND_t \) indicator random variables \( \tau_E \) and \( N \) indicator random variables \( \tau_x \). The first type random variables are 1 with probability \( 1/D_t \) and the second with probability \( O((D_{t+1} \log n/D_t)^{1/2}) \). For \( |V^*| \), (12) implies that any strategy may have variance bounds with total variance
\[ C_1 = \sum_{E \in H} O(1/D_t) + \sum_{x \in V} O((D_{t+1} \log n/D_t)^{1/2}) = O(N). \]

Thus Lemma 3.2 gives
\[ \Pr[|V^*| - (1 - p^*)N > (\log n)^{-2}N] \leq e^{-\Omega((\log n)^{-4}N)} = o(1), \]
and by (11)
\[
\Pr \left[ |V^*| \geq \left( 1 - e^{-\left( \frac{t}{n} \right) \log n} \right) \left( 1 + c_2(D_{t+1} \log n/D_t) \right) \left( 1 + (\log n)^{-2} \right) N \right] = o(1). 
\]

**Degrees.** For the expectations of $\deg^*(S)$, we first prove the following claim.

**Claim.** Let $x_1, \ldots, x_i \in V$, $1 \leq i \leq \binom{t}{k}$. Then, for an edge $E$ not containing any of $x_1, \ldots, x_i$,
\[
\Pr \left[ \{x_1, \ldots, x_i\} \subseteq V^* | E \in M \right] = \Pr \left[ \{x_1, \ldots, x_i\} \subseteq V^* \right] + O(D_{t+1}/D_t). 
\]  
(18)

That is, the two events $\{x_1, \ldots, x_i\} \subseteq V^*$ and $E \in M$ are almost independent.

**Proof.** Let $\mathcal{F}$ be the set of all edges that contain at least one of $x_1, \ldots, x_i$. Since $E \in M$ if and only if $E \subseteq X$ and $X \cap N_b(E) = \emptyset$, where $N_b(E) = \{E' : E' \neq E, |E' \cap E| \geq t\}$, the condition affects the event $\{x_1, \ldots, x_i\} \subseteq V^*$ only if $X$ contains $F \in \mathcal{F} \cap N_b(E)$ (by $E \subseteq X$) or $F' \in \mathcal{F} \setminus N_b(E)$ with $|F' \cap E'| \geq t$ for some $E' \in N_b(E)$ (by $X \cap N_b(E) = \emptyset$). For the former case, the amount of affection is
\[
\Pr [X \cap \mathcal{F} \cap N_b(E) \neq \emptyset] \leq \sum_{j=1}^{i} \sum_{x \in E} \sum_{F \supseteq x} \Pr [F \in X] = \sum_{j=1}^{i} \sum_{x \in E} \operatorname{codeg}(x, x)/D_t = O(D_{t+1}/D_t).
\]

For the later, the amount is at most
\[
\sum_{F' \in \mathcal{F} \setminus N_b(E)} \sum_{E' \subseteq N_b(E)} \sum_{|E' \cap F'| \geq t} \Pr [E', F' \in X] \leq \sum_{F' \in \mathcal{F} \setminus N_b(E)} \sum_{x \in E} \operatorname{codeg}(x, x')/D_t^2 = O\left( \frac{D_t D_{t+1}}{D_t^2} \right) = O\left( \frac{D_{t+1}}{D_t} \right).
\]

For a formal proof, let $X_1 = X \setminus (N_b(E) \cup \{E\})$ and $X_2 = X \setminus X_1$. Then, since $\Pr [X_1 = A, X_2 = B] = \Pr [X_1 = A] \Pr [X_2 = B]$ and
\[
\Pr [X_1 = A, X_2 = B | E \in M] = \begin{cases} \Pr [X_1 = A] & \text{if } B = \{E\} \\ 0 & \text{otherwise} \end{cases},
\]
we have
\[
\Pr \left[ \{x_1, \ldots, x_i\} \subseteq V^* \right] - \Pr \left[ \{x_1, \ldots, x_i\} \subseteq V^* | E \in M \right] \\
= \sum_{A, B} \left( \Pr [X_1 = A, X_2 = B] - \Pr [X_1 = A, X_2 = B | E \in M] \right) \mathbb{I}(A, B) \\
= \sum_{A, B} \Pr [X_1 = A] \Pr [X_2 = B] \mathbb{I}(A, B) - \mathbb{I}(A, \{E\}),
\]
where
\[
\mathbb{I}(A, B) = \begin{cases} 1 & \text{if } \{x_1, \ldots, x_i\} \subseteq V^* \text{ for } X = A \cup B \\ 0 & \text{otherwise}. \end{cases}
\]
(Strictly speaking, $\mathbb{I}(A, B)$ depends upon $W$ too, which, however, is irrelevant since $x_j \in W$ independently of any other events.) Clearly, $\mathbb{I}(A, \{E\}) = \mathbb{I}(A, \emptyset)$ and $\mathbb{I}(A, B)$ and $\mathbb{I}(A, \emptyset)$ differ only if $B$ contains $F \in \mathcal{F} \cap N_b(E)$, or $A$ and $B$ contain $F' \in \mathcal{F} \setminus N_b(E)$ and $E' \in N_b(E)$, respectively, such that $|F' \cap E'| \geq t$. 

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Lemma 4.2
\[
\mathbb{E}[\text{deg}\_s^*(S)] = (1 - p^*)^{(k\choose i)} - (i\choose s)\text{deg}\_s(S)\left(1 + O(D_{s+1}/D_s)\right),
\]
for all \( t \leq s \leq k - 1. \)

**Proof.** By the linearity of expectation,
\[
\mathbb{E}[\text{deg}\_s^*(S)] = \sum_{E \subseteq E} \Pr[x \in V^* \forall x \in \left(\binom{E}{t} \setminus \binom{S}{t}\right)].
\]

Let \( x_1, \ldots, x_i \) be the all vertices contained in \( \binom{E}{i} \setminus \binom{S}{i} \), \( i = (k\choose i) - (i\choose s) \). We will show by induction that
\[
\Pr[\{x_1, \ldots, x_i\} \subseteq V^*] = (1 - p^*)^i + O((j - 1)D_{t+1}/D_t)).
\]
The result follows from this since \((1 - p^*)^j = \Omega(1)\) and \(D_{t+1}/D_t = O(D_{s+1}/D_s)\).

The base case follows by (1). For \( j > 1 \), it is enough to show that
\[
\Pr[\{x_1, \ldots, x_{j-1}\} \subseteq V^*, x_j < M] = \Pr[\{x_1, \ldots, x_{j-1}\} \subseteq V^*] \Pr[x_j < M] + O(D_{t+1}/D_t).
\]

Since no two edges containing \( x_j \) are in (matching) \( M \), by (18), we have
\[
\Pr[\{x_1, \ldots, x_{j-1}\} \subseteq V^*, x_j < M] = \sum_{\substack{E \subseteq E \backslash \binom{S}{i} \forall 1 \leq i \leq j-1 \\text{\{}x_j \not\in X\text{\}}} \Pr[E \in M] \left( \Pr[\{x_1, \ldots, x_{j-1}\} \subseteq V^*] + O(D_{t+1}/D_t) \right)
\]
\[
= \sum_{\substack{E \subseteq E \\forall 1 \leq i \leq j-1 \\text{\{}x_j \not\in X\text{\}}} \Pr[E \in M] \left( \Pr[\{x_1, \ldots, x_{j-1}\} \subseteq V^*] + O(D_{t+1}/D_t) \right)
\]
\[
= \left( \Pr[\{x_1, \ldots, x_{j-1}\} \subseteq V^*] + O(D_{t+1}/D_t) \right) \sum_{\substack{E \subseteq E \\forall 1 \leq i \leq j-1 \\text{\{}x_j \not\in X\text{\}}} \Pr[E \in M].
\]

Since
\[
0 \leq \Pr[x_j < M] - \sum_{\substack{E \subseteq E \\forall 1 \leq i \leq j-1 \\text{\{}x_j \not\in X\text{\}}} \Pr[E \in M] \leq \sum_{l=1}^{j-1} \sum_{E \subseteq x_l, x_j} \Pr[E \in X] = O(D_{t+1}/D_t),
\]
the result follows.

We now present Paul’s strategy finding \( Y = \text{deg}^*(S) \) and its variance bounds. We say that Paul exposes or tries \( E \in H \) (resp. \( x \in V \)) if he asks to Carole if \( E \in X \) (resp. \( x \in W \)). If Paul exposes an edge \( E \) and the answer of Carole is ‘Yes’ (i.e. \( \tau_E = 1 \)), we call the edge a chosen edge. For a given set \( S \), write \( S \subseteq V \) if all \( l \)-subsets of \( S \) are in \( V \). Let \( E_S \) be the set of all edges \( E \) such that \( E \) contains \( S \) or \( |E \cap F| \geq t \) for some \( F \) containing \( S \). Paul’s first exposing rule is to expose an edge
in $\mathcal{E}_S$ which contains no common vertex with any of previously chosen edges. When Paul cannot expose any more by the first rule, he exposes all edges not in $\mathcal{E}_S$ and then all remaining edges in $\mathcal{E}_S$. For the precise definition of Paul’s strategy, let

$$\phi(E) := \sum_{x \in E} \text{codeg}_s(S, x) + \frac{1}{D_t} \sum_{E', |E'| \geq t} \sum_{x \in E'} \text{codeg}_s(S, x).$$

Paul orders all edges in $H$, say with ordering $\succ$, so that

$$E \succ F \implies \phi(E) \geq \phi(F). \quad (20)$$

This condition is technical but will play an important role later. Write $H = \{E^{(1)} \succ E^{(2)} \succ \cdots \succ E^{(|H|)}\}$. Paul’s rule is to expose the largest (according to $\succ$) non-exposed edge $E \in \mathcal{E}_S$ which contains no common vertex with any of previously chosen edges. We call edges exposed by this rule primary edges. If no such edge exists, Paul exposes all edges not in $\mathcal{E}_S$ and then all remaining edges in $\mathcal{E}_S$. We assume here that there is at least one edge not in $\mathcal{E}_S$. (Otherwise, we may add one not intersecting with any of $F \in e_S$ so that $\text{deg}_s^*(S)$ remains the same.) This assumption is not essential but will simplify notations (see e.g. (21)). Notice that the order of exposures depends only on Carol’s answers to primary edges. The order of exposing vertices is not relevant. Paul just fixes any order and exposes vertices according to the order after all edges are exposed. The order $\succ$ then can be naturally extended to be defined on all edges and vertices. This strategy has depth $r := |H| + |V|$.

Recall $U_x = \text{codeg}(S, x)$ and,

$$U_E = U^{(p)}_E I(E : \text{isolated in } X) + U^{(s)}_E,$$

where

$$U^{(p)}_E := \sum_{x \in E} \text{codeg}_s(S, x)$$

and

$$U^{(s)}_E(\tau) := \sum_{E' : \|E'\| \geq t \text{ isolated in } X'} \sum_{x \in E'} \text{codeg}_s(S, x).$$

We will apply Lemma 3.3 with the above $U_E$ and $U_x$.

Let $\tau = \tau^{(i)} = (\tau_1, \ldots, \tau_i)$ be a sequence Carole’s answers to Paul’s queries with $\tau_j = \tau_{E_j}$, $j = 1, \cdots, i$. (No vertex is exposed yet.) Notice that the edge $E_j$ is determined by $\tau_1, \cdots, \tau_{j-1}$. We write $U_j$, $U^{(p)}_j$ and $U^{(s)}_j$ for $U_{E_j}$, $U^{(p)}_{E_j}$, $U^{(s)}_{E_j}$, respectively. Define $T$ to be the last time when all exposing edges are in $\mathcal{E}_S$ and $T \land i = \min\{i, T\}$, i.e.,

$$T \land i = \max\{j \in [i] : \{E_1, \cdots, E_j\} \subseteq \mathcal{E}_S\}. \quad (21)$$

Clearly, $T \land i$ depend only upon $\tau_1, \cdots, \tau_{i-1}$.

**Lemma 4.3** There is a constant $\beta$ such that if $T \land i \leq i - 1$, then

$$\mathbb{E}[U_i|\tau_1, \ldots, \tau_{i-1}]^2 \leq \beta D_{s+1} \sum_{1 \leq j \leq T \land (i-1)} \tau_j \sum_{x \in E_j} \text{codeg}_s(S, x) \quad (22)$$

and if $T \geq i$, then

$$\mathbb{E}[U_i|\tau_1, \ldots, \tau_{i-1}]^2 \leq \beta D_{s+1} \phi(E_i). \quad (23)$$
Proof. Suppose \( T \leq i - 1 \). Then any edge exposed after \( i - 1 \) has non-zero effect only if it has a common vertex with at least one of \( E_1, \ldots, E_t \). In particular, there is no primary effect of \( E_i \). Thus we have
\[
U_i = U_i^s \leq \sum_{j \leq T \land \delta E_i \geq t} \tau_j \sum_{x \in E_j} \text{codeg}_x(S, x).
\]
Since \( |U_i^s| = O(D_{s+1}) \) (see (17)) and the above upper bound is determined by \( \tau_1, \ldots, \tau_{i-1} \) (since \( T \land i \leq i - 1 \)), we have
\[
E[U_i | \tau_1, \ldots, \tau_{i-1}]^2 = O\left( D_{s+1} E[U_i | \tau_1, \ldots, \tau_{i-1}] \right) = O\left( D_{s+1} \sum_{j \leq T \land \delta E_i \geq t} \tau_j \sum_{x \in E_j} \text{codeg}_x(S, x) \right).
\]
Suppose \( T \geq i \). Then, by the definitions \( T \), each chosen edge up to trial \( i - 1 \) contains no \( t \)-subset of \( E_i \). Hence
\[
U_i^s \leq \sum_{E' : E' \neq E_i} \tau_{E'} \sum_{x \in E'} \text{codeg}_x(S, x)
\]
with all \( E' \) in the sum must be exposed after \( E_i \), which gives us
\[
E[U_i^s | \tau_1, \ldots, \tau_{i-1}]^2 \leq O\left( D_{s+1} E \left[ \sum_{E' : E' \neq E_i} \tau_{E'} \sum_{x \in E'} \text{codeg}_x(S, x) \right] \right)
= O\left( \frac{D_{s+1}}{D_t} \sum_{E' : E' \neq E_i} \sum_{x \in E'} \text{codeg}_x(S, x) \right).
\]
Trivially, \( U_i^p = O(D_{s+1}) \) (see (16)) yields
\[
E[U_i^p | \tau_1, \ldots, \tau_{i-1}]^2 = O\left( D_{s+1} \sum_{x \in E_i} \text{codeg}_x(S, x) \right).
\]
Using \( (a + b)^2 \leq 2(a^2 + b^2) \), we finally have
\[
E[U_i | \tau_1, \ldots, \tau_{i-1}]^2 \leq 2 \left( E[U_i^p | \tau_1, \ldots, \tau_{i-1}]^2 + E[U_i^s | \tau_1, \ldots, \tau_{i-1}]^2 \right)
= O\left( D_{s+1} \left( \sum_{x \in E_i} \text{codeg}_x(S, x) + \frac{1}{D_t} \sum_{E' : E' \neq E_i} \sum_{x \in E'} \text{codeg}_x(S, x) \right) \right)
= O\left( D_{s+1} \phi(E_i) \right).
\]
Notice that the upper bound in (23) is not independent of \( \tau_1, \ldots, \tau_{i-1} \) since \( E_i \) is determined by those values. However, we really need an upper bound independent of \( \tau_{i-1} \) (cf. (9)?). For the sake of our ordering \( \succ \), such an upper bound can be obtained as follows. Since \( E_i \) is exposed by the first rule at trial \( i \), \( E^{(i)} \succeq E_i \), which yields
\[
\phi(E^{(i)}) \geq \phi(E_i),
\]
for all \( E_i \) determined by \( \tau_1, \ldots, \tau_{i-1} \).
Corollary 4.4 If $T \wedge i = i$, then

$$\mathbb{E}[U_i|\tau_1, \ldots, \tau_{i-1}]^2 \leq \beta D_{s+1} \phi(E^{(i)}) \ .$$

\[\square\]

For each vertex $x_i \in V$ exposed in trial $i > |H|$, (15) gives

$$\mathbb{E}[U_i|\tau_1, \ldots, \tau_{i-1}]^2 \leq D_{s+1} \text{codeg}_s(S, x_i) .$$

(24)

We define $C_1, \ldots, C_r$, which are are basically the sums of bounds obtained in Lemma 4.3, Corollary 4.4, and (17), and then prove the hypothesis (9) of Lemma 3.3. For $x_i \in V$ exposed in trial $i > |H|$, $C_i$ is a constant

$$C_i := D_{s+1} \sum_{j \geq i} \text{codeg}_s(S, x_j) .$$

Particularly,

$$C_{|H|+1} = \sum_{x \in V} \text{codeg}_s(S, x) = \deg_s(S) \left( \binom{k}{t} - \binom{s}{t} \right) \leq \binom{k}{t} D_s .$$

(25)

Set

$$C^{(0)} := \binom{k}{t} D_s .$$

For a sequence $(\tau_1, \ldots, \tau_i)$, let

$$C^{(p)}_i := \frac{\beta D_{s+1}}{D_t} |E_s| \sum_{j=i}^{|E_s|} \phi(E^{(i)}) ,$$

and

$$C^{(s)}_{i+2}(\tau_1, \ldots, \tau_i) := \frac{\beta D_{s+1}}{D_t} \sum_{E \neq \{E_1, \ldots, E_{i+1}\}} \sum_{j=1}^{T \wedge (i+1)} \sum_{E_j} \tau_j \sum_{x \in E_j} \text{codeg}_s(S, x) .$$

This is well defined since $T \wedge i + 1$ and $\{E_1, \ldots, E_{i+1}\}$ are completely determined by $\tau_1, \ldots, \tau_i$. By the definition of Paul's strategy and $T$, the chosen $(\tau_j = 1)$ edges up to trial $T$ are isolated so that for each $x \in V$ there is at most one $E_j$ with $\tau_j = 1$ and the edge $E_j$, if any, is counted at most $\binom{k}{t} D_t$ times in the sum

$$\sum_{E \in H} \sum_{j=1}^{T \wedge (i+1)} \sum_{x \in E_j} \tau_j \sum_{x \in E_j} \text{codeg}_s(S, x) .$$

Thus (25) yields

$$\sum_{E \in H} \sum_{j=1}^{T \wedge (i+1)} \sum_{x \in E_j} \tau_j \sum_{x \in E_j} \text{codeg}_s(S, x) \leq \binom{k}{t} D_t \sum_{x \in V} \text{codeg}_s(S, x) \leq \binom{k}{t}^2 D_t D_s .$$

This implies that for $C^{(1)} = \beta \binom{k}{t} D_{s+1} D_s$,

$$C^{(s)}_{i+2} \leq C^{(1)} .$$

(26)
Define

\[ C_{i+2}(\tau_1, \ldots, \tau_i) := \begin{cases} C_{i+2}^{(s)}(\tau_1, \ldots, \tau_i) + C^{(0)} & \text{if } T \leq i \\ C_{i+2}^{(p)} + C^{(1)} + C^{(0)} & \text{if } T \geq i + 1, \end{cases} \]

for \( i = 1, \ldots, |H| - 2 \). Denote also \( C_1 := (\beta + 1)C^{(0)} + 2C^{(1)} \) and \( C_2 := C_2^{(p)} + C^{(1)} + C^{(0)} \).

We now show that \( C_i \)’s satisfy (9). For \( |H| < i \leq r = |H| + |V| \), (24) yields

\[ p_1(1 - p)E[U_i|\tau_1, \ldots, \tau_{i-1}]^2 + C_{i+1} \leq D_{s+1} \text{codeg}_s(S, x_i) + D_{s+1} \sum_{j \geq i+1} \text{codeg}_s(S, x_j) = C_i. \]

Suppose \( 1 \leq i \leq |H| \). We will show that

\[ \frac{1}{D_t} E[U_i|\tau_1, \ldots, \tau_{i-1}]^2 + C_{i+1}(\tau_1, \ldots, \tau_{i-1}) \leq C_i(\tau_1, \ldots, \tau_{i-2}). \]

If \( T \leq i - 1 \), then (22) yields

\[
\frac{1}{D_t} \mathbb{E}[U_i|\tau_1, \ldots, \tau_{i-1}]^2 + C_{i+1}(\tau_1, \ldots, \tau_{i-1}) \\
\leq \frac{\beta D_{s+1}}{D_t} \sum_{j: 1 \leq j \leq T \land (i-1)} \tau_j \sum_{x \in E_j} \text{codeg}_s(S, x) + C_{i+1}^{(s)}(\tau_1, \ldots, \tau_{i-1}) + C^{(0)} \mathbb{P}(i \leq |H| - 1) \\
\leq \frac{\beta D_{s+1}}{D_t} \sum_{E: E \notin \{E_1, \ldots, E_{i-1}\}} \sum_{j: 1 \leq j \leq T \land (i-1)} \tau_j \sum_{x \in E_j} \text{codeg}_s(S, x) + C^{(0)} \\
= C_i^{(s)}(\tau_1, \ldots, \tau_{i-2}) + C^{(0)} \leq C_i(\tau_1, \ldots, \tau_{i-2}),
\]

where the second inequality is actually equality unless \( i = |H| \) in which case it uses (25), and similarly the last inequality is actually equality unless \( T = i - 1 \) in which case it uses (26). Suppose now \( T \geq i \). Then \( T \land (i - 1) = i - 1 \) and Corollary 4.4 give

\[
\frac{1}{D_t} \mathbb{E}[U_i|\tau_1, \ldots, \tau_{i-1}]^2 + C_{i+1}(\tau_1, \ldots, \tau_{i-1}) \leq \frac{\beta D_{s+1}}{D_t} \phi(E^{(i)}) + C_{i+1}^{(p)} + C^{(1)} + C^{(0)} \\
= C_i^{(p)} + C^{(1)} + C^{(0)}. 
\]

If \( i > 1 \) then \( C_i^{(p)} + C^{(1)} + C^{(0)} = C_i \). Suppose \( i = 1 \). Notice that

\[
C_{1}^{(p)} = \frac{\beta D_{s+1}}{D_k} \sum_{E \in H} \phi(E^{(1)}) \leq \frac{\beta D_{s+1}}{D_t} \sum_{E \in H} \phi(E) \\
= \frac{\beta D_{s+1}}{D_t} \sum_{E \in H} \sum_{x \in E} \text{codeg}_s(S, x) + \frac{\beta D_{s+1}}{D_t^2} \sum_{E \in H} \sum_{E' \subseteq E} \sum_{x \in E'} \text{codeg}_s(S, x). 
\]

In the first sum, each \( x \) is counted at most \( D_t \) times so that (25) gives

\[
\frac{\beta D_{s+1}}{D_t} \sum_{E \in H} \sum_{x \in E} \text{codeg}_s(S, x) \leq \frac{\beta D_{s+1}}{D_t} D_t \sum_{x \in V} \text{codeg}_s(S, x) \leq \binom{k}{t} \beta D_{s+1} D_s = \beta C^{(0)}. 
\]
In the second sum, each \(x\) is counted at most \(\binom{k}{t} D_t^2\) times, which along with (25) yields
\[
\frac{\beta D_{s+1}}{D_t^2} \sum_{E \in H} \sum_{E' | E' \cap E \geq 1} \sum_{x \in E'} \text{code}_g(S, x) \leq \left( \frac{k}{t} \right) \frac{\beta D_{s+1}}{D_t^2} D_t^2 \sum_{x \in V} \text{code}_g(S, x)
\]

\[
\leq \left( \frac{k}{t} \right)^2 \beta D_{s+1} D_s = C^{(1)}.
\]

Thus
\[
C_1^{(0)} + C^{(1)} + C^{(0)} \leq (\beta + 1)C^{(0)} + 2C^{(1)} = C_1.
\]

Since \(C_1 = O(D_{s+1} D_s)\), Lemma 3.3 gives for big enough constant \(c_3\),
\[
\Pr \left[ |Y - \mathbb{E}[Y]| > c_3(D_{s+1} D_s \log n)^{1/2} \right] \leq n^{-2s}.
\]

There are at most \(n^s\) \(s\)-subsets of \([n]\), this along with (11) and (19) implies that
\[
\Pr \left[ \exists S \text{ deg}_g^*(S) > \left( 1 - e^{-\binom{t}{k}} \right)^{\binom{t}{k}-1} \left( 1 + c(D_{s+1} \log n/D_s)^{1/2} \right) D_t \right] = o(1),
\]
for some large constant \(c > 0\) as well as
\[
\Pr \left[ \exists x \text{ deg}_g^*(x) < \left( 1 - e^{-\binom{t}{k}} \right)^{\binom{t}{k}-1} \left( 1 - c(D_{t+1} \log n/D_t)^{1/2} \right) (D_t - \Delta) \right] = o(1).
\]

For the last inequality of (4), let
\[
D_s^* = \left( 1 - e^{-\binom{t}{k}} \right)^{\binom{t}{k}-1} \left( 1 + c(D_{s+1} \log n/D_s)^{1/2} \right) D_t.
\]

Then
\[
\text{deg}_t^*(x) \geq \left( 1 - e^{-\binom{t}{k}} \right)^{\binom{t}{k}-1} (D_t - \Delta) - c(D_{t+1} D_t \log n)^{1/2}
\]
\[
\geq D_s^* - \left( 1 - e^{-\binom{t}{k}} \right)^{\binom{t}{k}-1} \Delta - 2c(D_{t+1} D_t \log n)^{1/2}
\]
\[
\geq D_s^* - (K \left( 1 - e^{-\binom{t}{k}} \right)^{\binom{t}{k}-1} + 2c) (D_{t+1} D_t \log n)^{1/2}.
\]

Notice that
\[
\left( \frac{D_{t+1} D_t}{D_{t+1} D_t} \right)^{1/2} \leq \left( \left( 1 - e^{-\binom{t}{k}} \right)^{1-\binom{t}{k}+1-\binom{t}{k}} \right)^{1/2} \leq \left( 1 - e^{-\binom{t}{k}} \right)^{1-\binom{t}{k}+t/2},
\]
and
\[
(K \left( 1 - e^{-\binom{t}{k}} \right)^{\binom{t}{k}-1} + 2c) \left( \frac{D_{t+1} D_t}{D_{t+1} D_t} \right)^{1/2} \leq (K + 2c \left( 1 - e^{-\binom{t}{k}} \right)^{1-\binom{t}{k}}) \left( 1 - e^{-\binom{t}{k}} \right)^{t/2}.
\]

We may take large enough \(c \geq c_1, c_2\) and satisfying for \(K = e^{3}\),
\[
(K + 2c \left( 1 - e^{-\binom{t}{k}} \right)^{1-\binom{t}{k}}) \left( 1 - e^{-\binom{t}{k}} \right)^{t/2} \leq K
\]
so that
\[
\text{deg}_t^*(x) \geq D_s^* - \Delta^*.
\]
Remark. Recently, we have learned that Vu obtained a slightly weaker bound of

\[ O\left(\frac{(\log n)^c}{n} (k-t)^3 (\frac{k}{t} - 1)\right), \]

which is a corollary of a more general result about certain hypergraphs.

References


