Triangulation and Embedding using Small Sets of Beacons

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Motivation: Internet latencies

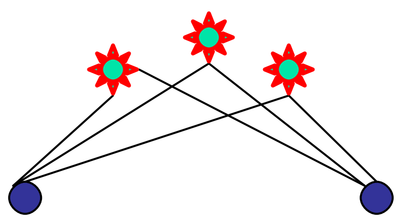
- Latency (round-trip time) is a notion of distance in Internet
  - distance matrix defined by latencies is almost a metric
- Two lines of research:
  - networking community studies the matrix of latencies
  - in FOCS/STOC community, lots of research on metrics
- Can we make a connection? fundamental disconnect:
  - theoretical studies: assume full access to distance matrix
  - distributed settings: only small #latencies can be measured
- We try to correct this discrepancy, in the context of ...

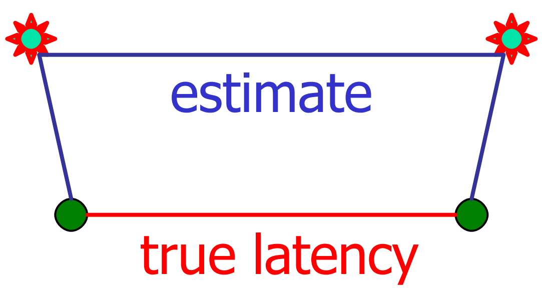
Estimating Internet latencies

- Many networking papers try to estimate latencies for many or most node pairs using only a small #measurements
  - one reliable measurement = e.g. min of 220 pings
- Our setting: large overlay network in Internet
  - P2P network, file-sharing system, distributed gaming app
- Applications:
  - find nearest neighbors (servers, file replicas, game buddies)
  - find farther neighbors (fault-tolerant storage)
  - proximity routing in P2P networks
  - two nodes can quickly estimate their latency

How to estimate latencies?

- IDMaps [Francis+ '01]: select small #nodes as "beacons"
  - users measure latency to beacons, not to one another
  - two nodes estimate latency via triangle inequality
- magic: relative error <1 on 90% node pairs
  - 900 random nodes, 15 beacons
  - relative error(x,y) = |x-y| / min(x,y)
- Similar approaches: [Hotz '94], [Guyton+ '95], [Kommareddy+ '01]

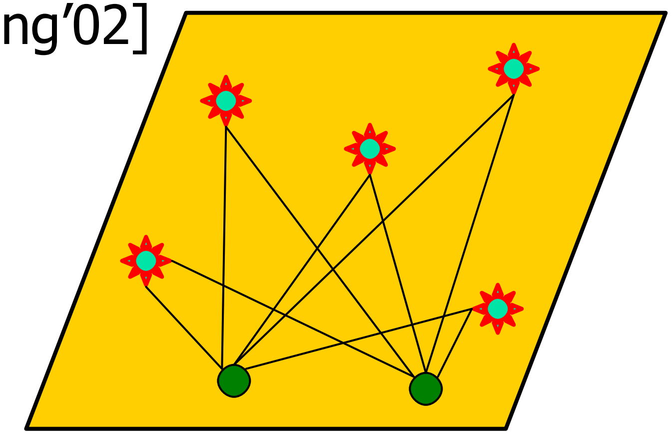




Estimate latencies via embedding

Global Network Positioning (GNP) [Ng+Zhang'02]

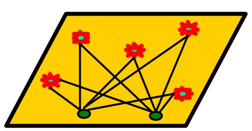
- select small #nodes as "beacons"
  - users measure latencies to beacons
- embed into low-dim Euclidian space
  - embed the beacons
  - embed non-beacons one by one



- magic: 90% node pairs are embedded with relative error <.5
  - 900 random nodes, 15 beacons, 7 dimensions
- lots of follow-up work: NPS [Ng+ '03], Vivaldi [Cox+ '04], Lighthouses [Pias+ '03], BigBang [Tankel+ '03], ICS [Lim+ '03], Virtual Landmarks [Tang+ '03], PIC [Costa+ '04], PALM [Lehman+ '04]

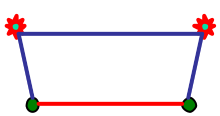
How can we explain the magic?

- How can we explain IDMaps and GNP theoretically?
  - assume that latencies form a metric
- Classical problem: embed into Euclidean space (or into L_p , $p \geq 1$)
  - [Bourgain '85], [Linial+ '95], [Rao '99], [Gupta '02], ...
  - full access to distance matrix (not suitable for networking apps)
- In beacon-based model, we get first provable guarantees for:



GNP-style embeddings into Euclidean space (and L_p , $p \geq 1$)

- constant distortion for most edges



IDMaps-style distance reconstruction

- upper and lower bounds that are close for most edges

Extra: L_p space

d-dimensional L_p space = d-dimensional space of reals with norm

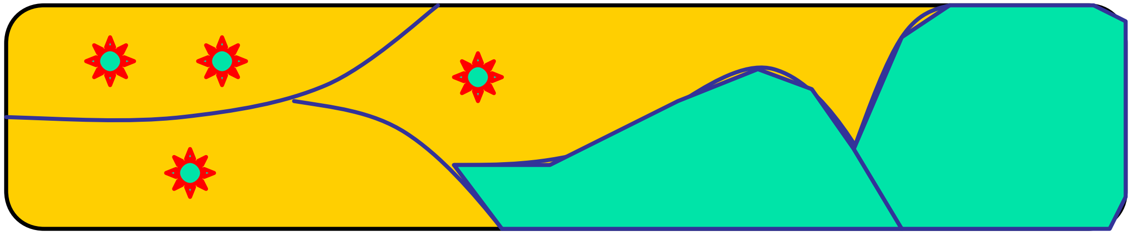
$$\|x-y\|_p = \sum_{i=1}^{d} |x_i - y_i|^p$$

- e.g. L_2 is the usual Euclidian norm

Relaxed guarantees (1/2)

Cannot estimate all distances with small #beacons

- partition nodes into k equal-size clusters, k>#beacons
- distance 0 within each cluster, 1 between clusters
- many clusters do not contain any beacons
- in these clusters distance to every beacon is 1
  ⇒ no way to tell which node lies in which cluster



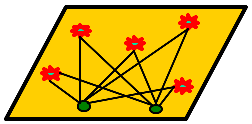
Relaxed guarantees (2/2)

- Allow "ε-slack": guarantees for all but *e*-fraction of node pairs
  - empirical results of GNP and IDMaps have ε-slack, too
  - new angle for theoretical work on embeddings
- This seems similar to property testing
  - similar goal: get global info using small #probes
  - guarantees with some form of slack
- However, there are differences:
  - "does property hold?" vs "embed/ reconstruct distances"
  - "change small fraction of distances" vs "ignore them"

Doubling metrics

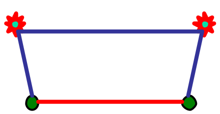
- In our framework, no good guarantees for arbitrary metrics
  - latencies are not arbitrary! need some restrictions ...
- Consider s-doubling metrics
  - Def any ball can be covered by s balls of half the radius
  - generalize low-dim Euclidean space: dimension ≈ O(log s)
  - suggested as good model for latencies, e.g. [Plaxton+ '97]
- Theoretical work: Assouad'83, [GKL'03], [KL'04], [LMN'04], ...
  - embeddings, nearest neighbor, TSP, distance labeling
  - embed into L_p, $p \geq 2$ with distortion at least $\Omega\left(\sqrt{\log n}\right)$
    - recall: if $.5 < d_{true} / d_{estimate} < 2$ then distortion < 4

Our results for doubling metrics



GNP-style embeddings into Euclidean space (and L_p , p≥1)

- assume s-doubling metric, allow ε-slack
- $O(s^5/\varepsilon)$ beacons ⇒ distortion $O(\log s/\varepsilon)$
- guarantees independent of #nodes
- recall that we cannot do this without ε-slack!



IDMaps-style distance reconstruction

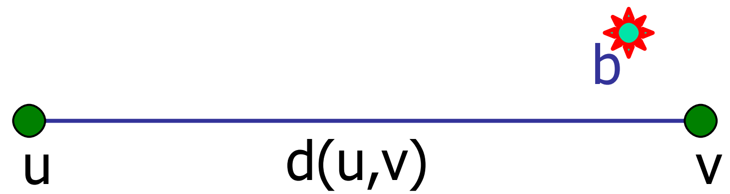
- estimate using triangle inequality
- $s^{O(\log 1/\delta)}/\varepsilon$ beacons ⇒ upper and lower bounds that are within factor $1+\delta$ for all but ε-fraction of edges

Overview

- ☑ Introduction
- ❑ IDMaps-style distance reconstruction
- ❑ GNP-style metric embedding
- ❑ "Fully distributed" approaches: low load on every node
- ❑ Extensions and open questions

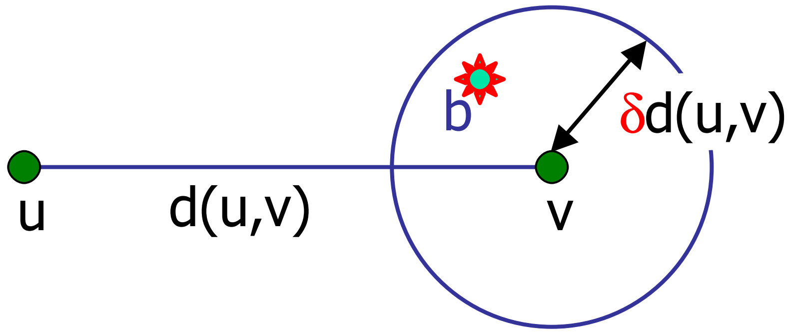
IDMaps-style distance reconstruction

- Rules: randomly select small #nodes as "beacons"
  - each node measures distances to all beacons
  - nodes (u,v) exchange their data and estimate d(u,v)
- Triangulation: a way to form such estimate via triangle inequality
  - (u,v), beacon b : $|d(u,b)-d(v,b)| \le d(u,v) \le d(u,b)+d(v,b)$
  - lower bound: $d^{-}(u,v) = \max_{\text{beacons } b} |d(u,b)-d(v,b)|$
  - upper bound: $d^{+}(u,v) = \min_{\text{beacons } b} d(u,b)+d(v,b)$

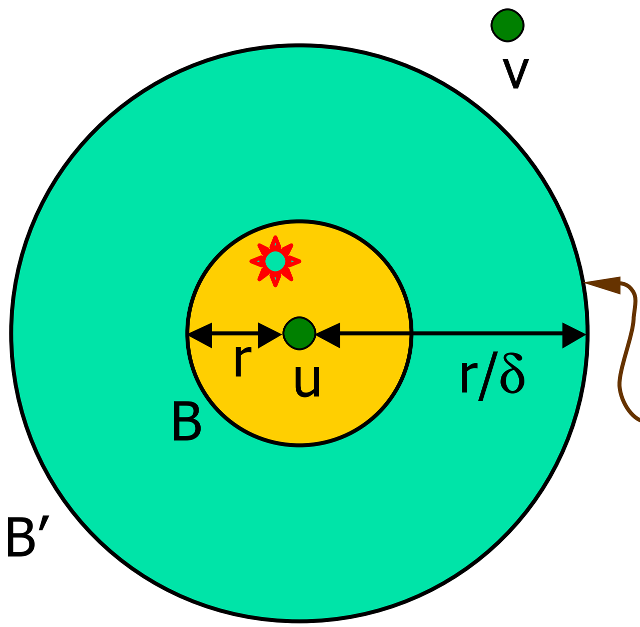


Triangulation: proof (1/3)

- Want the bounds to be close, for most node pairs
- Def node pair (u,v) is δ-good if some beacon lies within distance $\delta d(u,v)$ from u or v
  - then $d^{+}(u,v) / d^{-}(u,v) \leq 1+3\delta$
- It suffices to show that most node pairs are δ-good



Triangulation: proof (2/3)



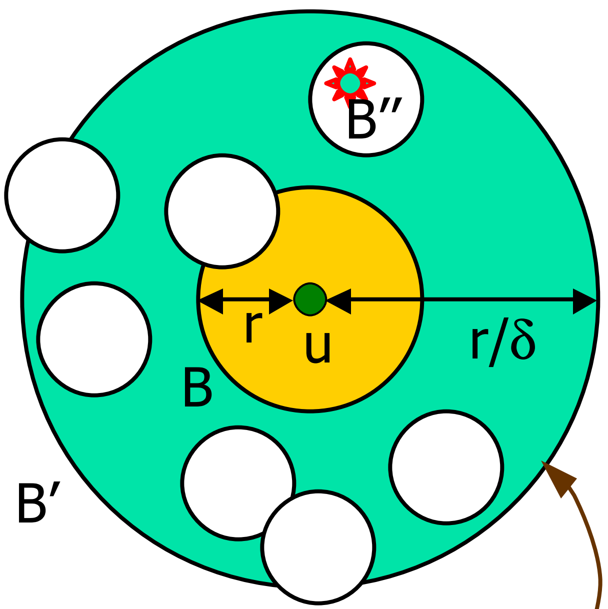
<u>Thm</u> Use $s^{O(\log 1/\delta)}/\varepsilon$ beacons $\Rightarrow$ w.h.p. all but ε-fraction of node pairs are δ-good

- assume s-doubling metric

Fix node u, consider pairs (u,v)
B = smallest ball around u with εn nodes
r = radius of B
B′ = ball around u of radius r/δ

Since $|B| \geq \varepsilon n$, B contains a beacon w.h.p.
$\Rightarrow$ pairs (u,v), $v \notin B'$ are δ-good
What about pairs (u,v), $v \in B'$?

Triangulation: proof (3/3)



B = smallest ball around u that contains ϵn nodes

- Doubling property ⇒ B′ can be covered by k =f(s,ε,δ) small balls of radius δr
- Fix some small ball B″ , consider pairs (u,v), v∈B″
  - if B″ contains >εn/k nodes, then w.h.p. it contains a beacon, so these pairs are δ-good
  - else ignore these pairs
- only ≤ϵn node pairs ignored for each u

Theorem proved

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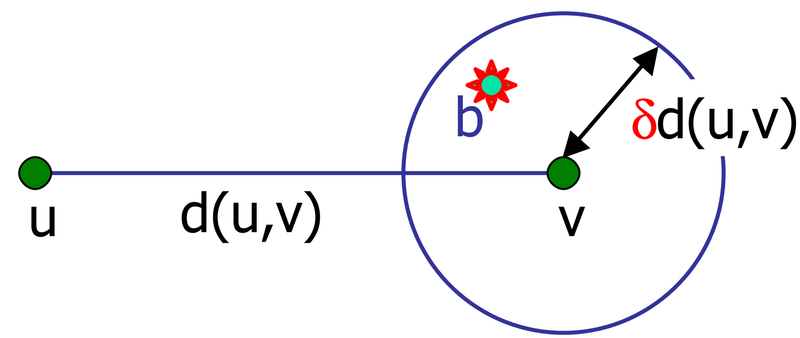
- ❑ GNP-style metric embedding
- ❑ "Fully distributed" approaches: low load on every node
- ❑ Extensions and open questions

GNP-style metric embedding (1/3)

- Rules: embed s-doubling metric into L_p , p ³ 1
  - randomly select k nodes as "beacons"
  - coordinates of node = fn(its distances to beacons)
  - allow ε-slack; want k, dimension and distortion to be fn(s,ε)
- Pick k so that all but ε-fraction of node pairs are Θ(1/log k)-good
  - recall: node pair (u,v) is δ-good if some beacon lies within distance δd(u,v) from u or v
  - here δ = Θ(1/log k)



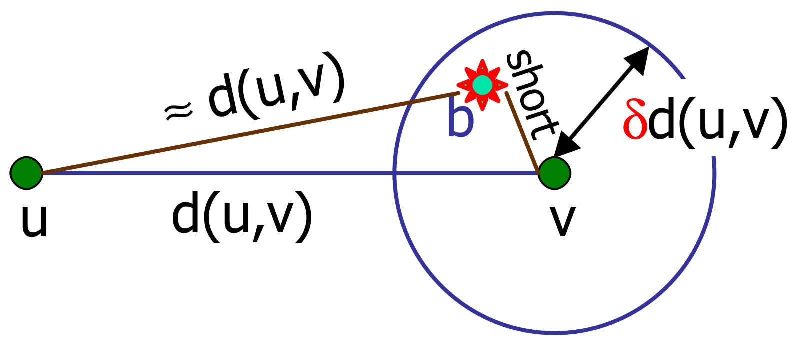
GNP-style metric embedding (2/3)

Claim Suppose distortion on distances to beacons is $D=O(\log k)$. Then distortion on every δ-good node pair is $O(D)$.

Proof sketch:

- set $\delta=1/3D$; it follows that $d_{embed}(v,b) \leq d_{embed}(u,b) / 2$
- $d(u,v) \approx d(u,b)$ both for true and embedded distances
- distortion on uv = O(distortion on ub) = O(D)

claim proved



GNP-style metric embedding (3/3)

- Thm distortion O(log k) on all but ε-fraction of node pairs
- It suffices to provide an embedding such that distortion on distances to beacons is at most O(log k)
- Our algorithm:
  - let S be the set of all beacons
  - for each i select $\Theta(k)$ random subsets S_{ij} of S of size 2^i
  - ij-th coordinate of node u = distance from u to S_{ij}
- This is similar to the algorithm of [Linial+ '95], except
  - they use only $\Theta(\log k)$ sets S_{ij} for each i
  - their proof bounds distortion only between the beacons, whereas we bound distortion for all node-beacon pairs

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Fully distributed approaches (1/4)

- Informal def fully distributed algorithm: low load on all nodes
- Recent trend in networking: fully distributed analogs of GNP
  - Vivaldi [Cox+ '04], Lighthouses [Pias+ '03], PIC [Costa+ '04], PALM [Lehman+ '04]
  - embed Internet latencies into low-dim Euclidean space
  - use distributed heuristics, with results similar to GNP
- Their approach:
  - each node selects k random nodes as neighbors
  - only distances to neighbors are measured
  - my set of neighbours is different from yours!

Fully distributed approaches (2/4)

- Basic question: Vivaldi only looks at distances to neighbors. Suppose these distances are embedded with small distortion. Any guarantees on the rest of the node pairs?
  - each node selects k neighbors uniformly at random
  - assume doubling metric, allow ε-slack, want small k
- Thm If distances to neighbors are embedded with distortion D then for all but ε-fraction of node pairs distortion is $O(D)$
  - essentially $k = (D \log n)^{\text{constant}}$

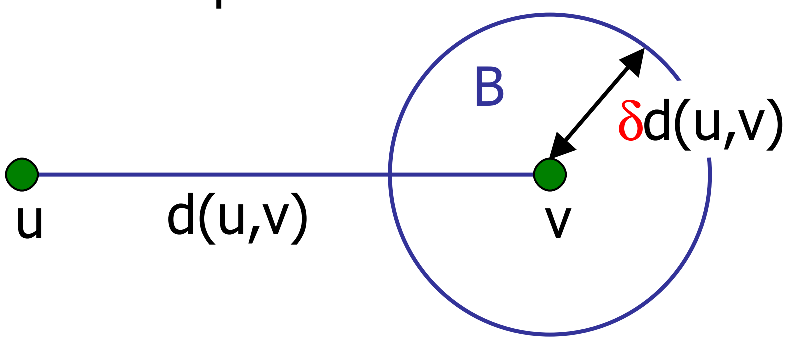
Fully distributed approaches (3/4)

Overview of the proof:

- assume distortion D on distances to neighbors
- all but ε-fraction of node pairs are δ-good, $\delta = 1/\Omega(D \log n)$

  def pair (u,v) is δ-good if ball B of radius δd(u,v) around u or v
  ~~has enough nodes so that it contains a beacon~~
  has enough nodes so that every node has ≥3 neighbours in B
- distortion O(D) on each δ-good node pair

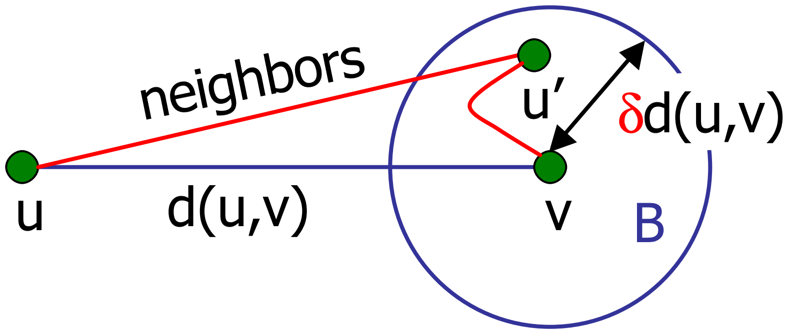
theorem proved



Fully distributed approaches (4/4)

Claim distortion O(D) on each δ-good node pair (proof sketch)

- recall: every node has ≥3 random neighbours in B
- set of neighbour pairs induces an expander on B
  ⇒ this expander contains u′v-path P with O(log n) hops
- in the embedding, each hop is distorted by at most D, so
  length(P) ≤ D×O(log n)×diam(B), much shorter than edge uu′
- d(u,v) ≈ d(u,u′)
  true and embedded
- distortion on uv
  = O(distortion on uu′)
  = O(D), claim proved



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- ☑ IDMaps-style triangulation
- ☑ GNP-style metric embedding
- ❑ "Fully distributed" approaches
  - ☑ general justification
  - ❑ triangulation and embedding with provable guarantees
- ❑ Extensions and open questions

Fully distributed triangulation (1/4)

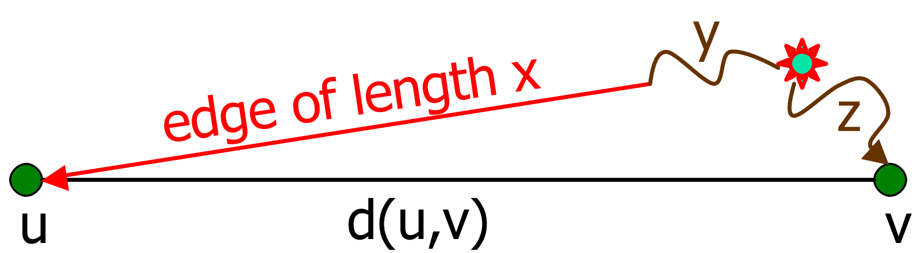
Overview of the algorithm:

- each node selects k neighbors, measures distances to them
  - my set of neighbors is different from yours!
- each node elects itself as a beacon with probability $\Theta(k/n)$
  - $\Theta(k)$ beacons total in the network
- if elected, beacon announces itself to network via gossiping
  - gossip to neighbors only; protocol is crucial (details coming ...)
- nodes receive announcements and estimate distances to beacons without measuring them directly (details coming ...)
- two nodes (u,v) exchange their estimates and estimate d(u,v)

Fully distributed triangulation (2/4)

How do nodes estimate distances to beacons?

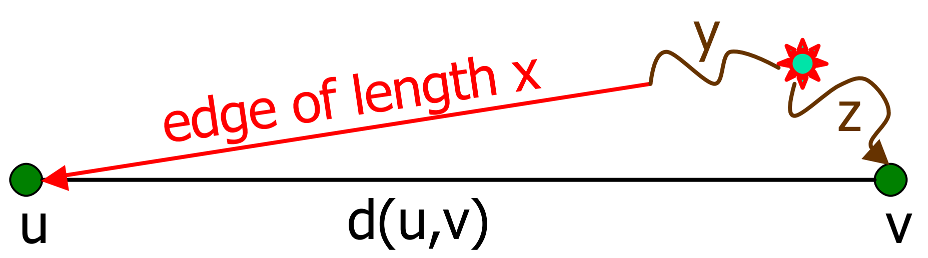
- node u deduces that d(u,) lies between x–y and x+y
- node v deduces that d(v,) lies between 0 and z
- nodes (u,v) exchange their bounds and deduce that d(u,v) lies between x–(y+z) and x+(y+z)
  - upper and lower bounds on d(u,v), like in beacon-based triangulation



Fully distributed triangulation (3/4)

Thm If $k = (D \log n)^{\text{constant}}$ then for all but ε-fraction of node pairs there will be a beacon such that $(y+z) < x/2D$

- then upper and lower bounds on $d(u,v)$ are within factor $1+1/D$

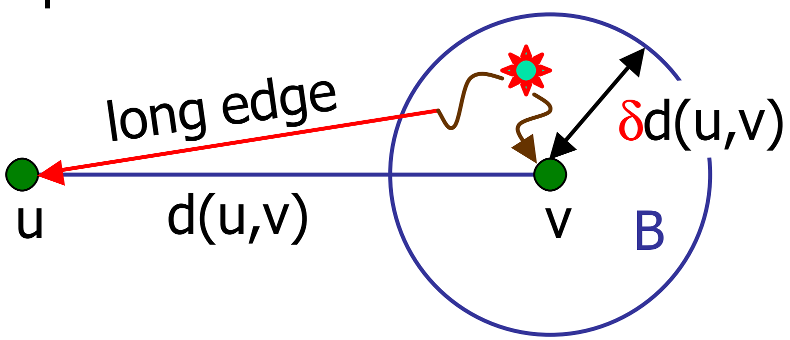


Fully distributed triangulation (4/4)

Proof sketch:

- all but ε-fraction of node pairs are δ-good, δ=1/Ω(D log n)
  - pair (u,v) is δ-good if ball B of radius δd(u,v) around u or v has enough nodes so that every node has ≥3 neighbours in B
- ball B contains a beacon; desired neighbor-to-neighbor paths exist since the set of neighbour pairs induces an expander on B
- gossiping protocol finds such paths (lots of details omitted)
- ⇒ for δ-good pairs the bounds are OK

theorem proved



Fully distributed metric embedding

Overview of the algorithm [Slivkins SODA'05]:

- run our fully distributed triangulation algorithm
  - compute upper bound d^+ on distances to beacons
- run our GNP-style embedding on same beacon set
  - for distances to beacons, use d^+ instead of d
  - analysis is more complicated since d^+ is not a metric

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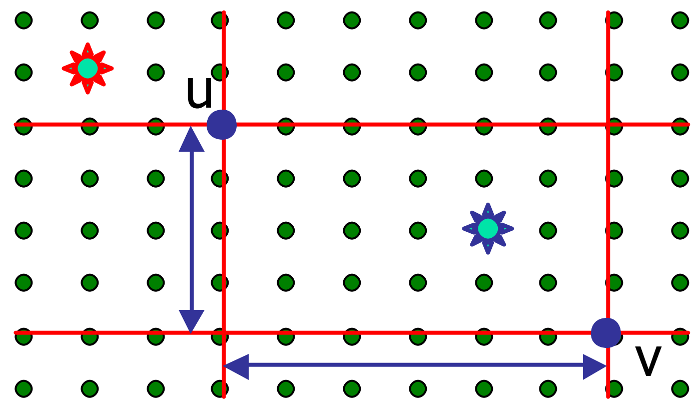
Strongly doubling metrics

- Def ball of twice the radius has at most s times many points
  - generalize grids in low-dimensional Euclidean space
  - [Karger+Ruhl ‘02], [Krauthgamer+Lee ‘03], [Bartal+Mendel ‘04], ...
- Thm embedding into L_p with gracefully degrading distortion:
  ¾ of node pairs embedded with distortion Δ = O(log s),
  ¾ of the remaining node pairs – with distortion 2Δ,
  ¾ of the remaining node pairs – with distortion 3Δ, ...
- Thm simple beacon-based embedding into L_p
  - select k nodes at random as “beacons”, embed them
  - embed each node at the nearest beacon

Triangulation in arbitrary metrics

- Lower bound d^- could be entirely useless
  - e.g. in uniform metric d^- =0 for all node pairs
  - recall: $d^-(u,v) = \max_{\text{beacons } b} |d(u,b) - d(v,b)|$
- Upper bound $d^+(u,v) \leq 3d(u,v)$
  - with ε-slack, $f(\varepsilon)$ beacons suffices
  - recall: $d^+(u,v) = \min_{\text{beacons } b} d(u,b)+d(v,b)$
- This factor of 3 is tight for triangulation
  - counterexample: complete bipartite graph
  - moreover, it is tight for any beacon-based estimate

Perfect triangulation: upper bound matches lower bound



- L_1 grid: $d(u,v) = X_{offset} + Y_{offset}$
- d^- = |d(u, ) – d(v, )|= d(u,v)
- d^+ = |d(u, )+ d(v, )|= d(u,v)
- Claim Select f(ε) beacons u.a.r. Then $d^-(u,v) = d^+(u,v) = d(u,v)$ for all but ε-fraction of pairs (u,v).
- Thm This extends to dense point sets in d-dim L_1 :
  - minimal distance between any two points is 1
  - all coordinates lie in $[0, kn^{1/d}]$, k = density parameter
  - f(ε,k,d) beacons suffices: #beacons independent of #nodes

Open questions

How can we extend our results to general metrics?

- Embed into L_p , p ³1 with ε-slack and distortion $f(\varepsilon)$?
  - recall: works for doubling metrics
- How well can you embed an arbitrary metric
  - using only distances to beacons?
  - via a fully distributed algorithm?
- Converse questions on beacon-based triangulation
  - good guarantees (e.g. $d^+/d^- \leq 1+\delta$) imply what?
- On-going follow-up work
  - [Chan, Dhamhere, Gupta, Kleinberg, Slivkins]