Probabilistic Relational Verification for Cryptographic Implementations


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Abstract

In the form of tools like EasyCrypt, relational program logics have been used for mechanizing formal proofs of various cryptographic constructions. With an eye towards scaling these successes towards end-to-end security proofs for implementations of distributed systems, we present RF*, a new extension of F*, a general-purpose higher-order stateful programming language with a verification system based on refinement types.

The distinguishing feature of RF* is a new probabilistic, relational Hoare logic, formalized in Coq—the first such logic for a higher-order, stateful language. We prove the soundness of this logic against a new denotational semantics for RF*, in contrast to prior operational formalizations of F*. Through careful language design, we adapt the F* typechecker to generate both classic and relational verification conditions, and to automatically discharge their proofs using an SMT solver. Thus, we are able to benefit from the existing features of F*, including, for example, its abstraction facilities that support modular reasoning about program fragments. We evaluate RF* experimentally by programming a series of cryptographic constructions and protocols, and by verifying their security properties, ranging from information flow to unlinkability, integrity, and privacy.

1. Introduction

Many fundamental notions of security go beyond what is expressible as a property of a single execution of a program. For example, noninterference [17], the property underlying information-flow security, relates the observable behaviors of two program executions. In addition to describing multiple executions of a program, security properties must often account for probabilistic behaviors. For instance, simulation-based properties relate a program to an imperfect simulation of it, characterizing the probability of failure, while the security of many cryptographic constructions is specified as a bound on the winning probability of an adversary in a probabilistic experiment.

Recognizing the importance to computer security of such hyperproperties [13], researchers have developed a range of program analyses and verification tools for proving relations between two or more programs, or two or more executions of the same program. In this context, one domain that has seen particularly striking advances is code-based provable security, an emerging approach to cryptography. For example, EasyCrypt [5] is a tool that implements a relational Hoare logic [7] over probabilistic imperative programs. The logic is able to justify common patterns of probabilistic reasoning about hyperproperties used in cryptographic proofs, including observational equivalence arguments, equivalence up to failure events, and reduction steps. As such, EasyCrypt provides a machine-checked framework for proving the security of cryptographic constructions in the computational model, and, over the last two years, has been used for verifying public-key encryption and signature schemes, modes of operation and hash function designs.

These advances, among others, raise the prospect of a new class of verifiably secure systems: those that are proven secure based on standard, computational assumptions about cryptography (such as the existence of one-way functions) and whose verification encompasses all aspects of the system implementation. Still, the formalisms and tools in the state of the art do not quite suffice to achieve this goal. For example, EasyCrypt is not designed for building, verifying and deploying systems. Specifically, the input language of EasyCrypt is an extension of a simple, imperative language with random assignments and procedure calls, and does not provide the commodities of a general-purpose programming language for developing realistic implementations. More fundamentally, EasyCrypt does not provide modular reasoning principles that allow probabilistic guarantees to be lifted to more abstract specifications that can be reused elsewhere in a program analysis.

On the other hand, another strand of research into code-based security has emerged. Several research groups have put non-relational, program verification systems for general purpose languages (ranging from C to ML) to use in proving properties of cryptographic implementations [8, 15, 16, 29]. Based on security assumptions on cryptographic libraries, these systems have been used to carry out modular proofs (to varying extents) on implementations at a relatively large scale. For example, over the last two years, F* [29], a dialect of ML, has been used for verifying over 50,000 lines of code, including implementations of multi-party sessions, web-browser extensions, zero-knowledge protocols, a full-fledged implementation of the Transport Layer Security 1.2 (TLS) standard, and the F* typechecker itself.

RF*: end-to-end security of cryptographic implementations In an effort to scale code-based security towards end-to-end security proofs of system implementations, this article presents a new language, RF*, which integrates within F* an expressive system of relational refinements to support fine-grained reasoning about probabilistic computations. Through careful language design, we are able to use the relational features of RF* in smooth conjunction with the existing features of F*, allowing the large corpus of already-verified F* code to be reasoned about effectively when used in a relational context. As such, our work opens the door to complet-
ing the certification of security proofs of critical pieces of Internet infrastructure, such as the implementation of TLS (of which large parts are already verified in F* ) using RF* . Technically, this paper makes three broad contributions, discussed next.

1. A probabilistic relational logic for higher-order stateful programs: using the Coq proof assistant, we formalize \( \lambda_0 \), a lambda calculus with references, random sampling, and unbounded recursion. We give it a denotational semantics, including both a standard set-theoretical interpretation, as well as a probabilistic, relational interpretation over pairs of store-passing functions.

We develop a type system for \( \lambda_0 \), and prove it sound with respect to the probabilistic, relational interpretation. This type system forms a logic for \( \lambda_0 \), the first such logic for higher-order, stateful programs. (§3)

2. The design and implementation of RF*: the logic of \( \lambda_0 \) forms the basis of the design of RF*, an extension of F*. We show how to encode the relational types of \( \lambda_0 \) within a new relational state monad, RDST, encoded in F*.

We provide a type inference algorithm for RDST in the form a weakest pre-condition calculus that computes relational verification conditions. Proofs of these verification conditions can be discharged automatically by the RF* typechecker and the Z3 [41] SMT solver. (§4)

3. An experimental evaluation of RF*: we demonstrate the expressiveness of RF* through a representative set of examples, starting from simple (non-probabilistic) information flow, and gradually moving towards advanced cryptographic models and systems. To date, we have used RF* to automatically verify a total of around 1,400 lines of code for a variety of relational properties, ranging from termination-insensitive noninterference to various indistinguishability-based properties for encryption, besides others. Several of our examples make essential use of both higher-order and stateful features of RF*, emphasizing the necessity of the \( \lambda_0 \) logic for practical security verification. (§2 and §5)

The \( \lambda_0 \) theory formalized in Coq, a compiler download for RF*, and all the example programs mentioned in this paper are available from http://research.microsoft.com/fstar.

2. Programming with relational refinements

We start by describing RF* informally through a series of examples, beginning with a brief introduction to F* itself, and then focusing on the main new feature in RF*, i.e., relational refinement types.

2.1 From classic to relational refinements

F* is a call-by-value higher-order programming language with primitive state and exceptions, similar to ML, but with a more expressive type system based on dependent refinement types. Refinement types are written \( \text{xt}\{\phi\} \) where \( \phi \) is a logical formula. For instance, the code fragment below defines a refined type for non-negative integers, then for integers modulo a prime number \( p \):

\[
\begin{align*}
type nat &= \text{nint} \{ 0 \leq n \} \\
type p &= \text{nnat} \{ n < p \} \\
let n \mod p &= 97 \\
let n \mod p &= 73
\end{align*}
\]

Typechecking F* programs involves logical proof obligations, which are delegated to the Z3 SMT solver. For instance, to check that \( n \) has type \( \text{mod} \ p \), the F* typechecker emits the proof obligation \( p = 97 \implies 73 < p \), which is easily discharged by Z3.

Type safety means that, whenever an expression \( e \) with type \( \text{xt}\{\phi\} \) reduces to a value \( v \), then \( v \) satisfies the formula \( \phi[n/x] \). The type system also provides structural subtyping. For instance, \( \text{nint} \) is a subtype of \( \text{int} \), and \( \text{mod} \ p \) is a subtype of \( \text{mod} \ q \) when \( p \leq q \). These subtyping relations are automatically proved and applied by F*.

Refinements can be combined with dependent function types, \( \text{xt} \rightarrow \text{t} \), where the formal parameter \( \text{xt} \) is in scope in the co-domain \( \text{t} \). We also use dependent pairs \( (\text{xt} \times \text{t})' \), where the name \( x \) of the first component is in scope in the type \( \text{t}' \) of the second.

For instance, we may write addition modulo as follows (where the refinement braces bind tighter than the arrow):

\[
\text{val add: } p:\text{nint} \rightarrow x:\text{mod} \ p \rightarrow y:\text{mod} \ p \rightarrow x\mod p \{ z = (x+y) \mod p \}
\]

\[
\begin{align*}
\text{let add } p \times x \times y &= \text{let } s = x + y \text{ in if } s < p \text{ then } s \text{ else } s - p
\end{align*}
\]

F* also provides primitive support for programming with state. For example, one may write \( \text{let incr } i = i : = i + 1 \). By combining refinements with references one can express invariants on the program state, e.g., \( \text{ref nat} \) is the type of mutable location containing a non-negative integer. To describe more precise properties of effectful programs, F* provides more advanced mechanisms, including a monadic mode [28], where one can reason about programs using variants of the Hoare state monad of Nanaveksi et al. [22] together with McCarthy’s select/update theory for modeling the heap [21]. For example, one can give \( \text{inc} \) a specification of the form: \( \text{ref nat} \rightarrow ST(\text{Ah,True}) \) unit (\( h \) \( h'\cdot h'' = \text{Upd} \ h \ (i + \text{Sel} h i) \)), where \( ST \) \( pre \) \( post \) can be understood as the state-passing function \( \text{hheap}[\text{pre} h] \rightarrow (\text{xt} \times \text{h'heap}[\text{post} h h']) \) although, in reality, F* provides primitive support for state.

That is, the type of \( \text{inc} \) states a trivial pre-condition on the input heap \( h \), and a post-condition indicating that the final heap \( h'' \) differs from \( h \) at the location \( i \), which is incremented. F* provides type inference in the form of a higher-order weakest pre-condition calculus to help ease the burden of writing such precise specifications.

RF* extends F* with relational refinements: every type can (also) be decorated with a relational formula, placed within braces \( \{ \cdot \cdot \cdot \} \), that specifies a joint property on pairs of values. Relational formulas can refer to the left and right value of every program variable in scope, using the projections \( L \) and \( R \), respectively; projections extend naturally to arbitrary formulas. Intuitively, for deterministic programs, type safety means that, whenever we obtain two results \( v_L \) and \( v_R \) by evaluating an expression \( e: \text{xt}\{\phi\} \) in two contexts that provide well-typed substitutions for \( e \)'s free variables, then the formula \( \phi[v_L/L][v_R/R] \) is valid. More generally, instead of considering two executions of the same program, RF* allows proving relations between pairs of programs, i.e., we relate \( e_0 \) and \( e_1 \) at a (relationally refined) type using \( e_0 \sim e_1 : t \). Indeed, we write \( t \) as a shorthand for \( e_0 \sim e_1 : t \).

We start with a few simple examples. Take the expression \( e \) to be \( z - z \); we can give it the type \( \text{xt}\{L x = R x\} \), meaning that for any pair of substitutions \( \sigma_L \) and \( \sigma_R \), evaluating \( \sigma_L(x) \) yields the same result as evaluating \( \sigma_R(x) \). Relational refinements can also be used to describe properties beyond equivalence, as illustrated by the type \( \text{xt}\{y\text{int}\{L x \leq R x \Rightarrow L y \leq R y\}\} \), of monotonic functions. To describe a \( k \)-sensitive integer function with respect to metric dist, we write \( \text{xt}\{y\text{int}\{DIST}(L y)(R y) \leq k \cdot \text{dist}(L x)(R x)\} \).

RF* can automatically check (by subtyping) that a function like \( \text{fun } x \rightarrow k \cdot x \) is both monotone and \( k \)-sensitive for \( k \geq 0 \).

Relational refinements are strictly more expressive than plain refinements: one can encode any plain refinement \( \{ \phi \} \) as the relational refinement \( \{ \{ L \phi \land R \phi \} \} \) that independently specifies \( \text{left} \) and \( \text{right} \) properties for instance, the type \( \text{nat} \) is automatically desugared to \( \text{nint} \{ 0 \leq L n \land 0 \leq R n \} \). Pragmatically, this enables us to mix property refinements and relational refinements in our concrete syntax, and to import any refinement-typed F* library in relational mode by applying the encoding. When authoring programs with specific non-classical relational properties in mind, one need issue only a single compiler directive (aka a pragma) to switch the verifier to relational mode. We contend that the resulting language, RF*, brings relational program verification out of the
domain of tools applied to small fragments of pseudocode with interactive proofs, to a practical programming language suitable for small- to medium-scale systems implementations.

2.2 Information flow

Relational refinements can be used to provide a semantic characterization of noninterference-based termination-insensitive information flow controls [27]. Whereas standard type-based information flow controls resort to ad hoc syntactic mechanisms to conservatively determine when a program’s observable outputs may depend on its secret inputs, RF* can directly verify the target equivalences. This section provides several examples.

If an expression e that computes over some secret information can be given the type eq a = x : a { L x = R x }, then its result can be released safely to an information flow adversary, since the executions of e reveal no information about the secrets. Capturing the intuition from label-based information flow type systems, the type eq a (the type of values that are “equal on both sides”) is the type of low-confidence values, so we also call it low a. In contrast, hi a is just an alias for a—their values may differ on either side.

Using these types, we can write programs like fun x → x → x and give them information flow types like hi int → low int. More interestingly, we can combine such types with logical properties to capture more general policies. For example, a plausible confidentiality policy for credit card numbers conceals all but their last four digits, as implemented below.

```plaintext
val last_four : n hi int → string \{ \{ L n = R n \} \% 10000 \implies L s = R s \}
let last_four n = "********" \% int2string \{ n \% 10000 \}
```

Tracking leaks via control dependences (aka implicit flows) is a characteristic feature of information flow type systems. To illustrate how RF* reasons about implicit flows, consider the program fun b → if b then e else e' . In order to give this program the type hi bool → low a, we need to analyze four cases that arise from applying this function twice to arbitrary boolean arguments L b and R b, and prove that the results in all cases are the same. The four goals are (1) e : low a, under the assumption that L b = R b = true; (2) e' : low a, under the assumption that L b = R b = false; (3) e ~ e': low a, assuming L b = true and R b = false; and (4) e' ~ e : low a, assuming L b = false and R b = true.

Our proof rules for relating two values v0 and v1 are relatively simple. Proving v0 ~ v1 : t (which for base types involves simply showing that both v and v' have type t), and then proving \phi (v0 \% L x \% t \% L x \% L x), So, RF* can easily prove (fun b → if b then e else e') : hi bool → low int, or (fun x → if x = 0 then e else e') : hi int → low int even though, syntactically, those functions branch on confidential values.

For expressions, particularly those that have side effects, the problem is harder. Our strategy is to adapt the Hoare monad ST pre post provided by F* to a relational version called RST, where RST pre post can be seen as the type shown below:

```
(h : heap)(\{ pre (L h) (R h) \})
→ (k : k hi h : heap)(\{ post (L h) (R h) (L x) (R x) (L h') (R h') \})
```

This is the type of pairs of functions that, when run in a pair of input heaps L h and R h satisfying the 2-place relational pre-condition predicate pre, may diverge, but if they both converge, yield results L x and R x and output heaps L h' and R h' that satisfy the 6-place relational post-condition predicate post.

Using the RST monad (and its associated weakest pre-condition calculus), we can write the following program:

```plaintext
val f : x ref int → b bool → RST (λ _ True) unit post \{ where post h0 h1 _ _ h'0 _ h1' = L x = R x ∧ h0@h1 ⇒ h'0 = h1' \}
  let f x b = if b then x := 1 else x := 1
```

RF* infers a weakest pre-condition predicate transformer for this program, then checks that it is consistent with any programmer supplied annotation (the annotation is optional for loop-free programs). The pre-condition of f states that it can be run in any pair of heaps, while its post-condition ensures that if f is applied twice to the same references in the same heaps, then regardless of the boolean argument, the resulting heaps are also the same, i.e., the type reveals that f does not leak information despite the side-effect control dependent on the secret boolean.

More complex programs, for example those that may leak information via side-effects based on aliasing, can similarly be verified.

```plaintext
val g : x ref int → y ref int → b bool → RST (λ _ True) unit post \{ where post h0 h1 _ _ h'0 _ h1' = L x = R y ∧ R x ∉ R y \implies Sel h0 (L y) = Sel h1 (R y) \}
  let g x y b = if b then x := 1; y := 1 else y := 1; x := 0
```

The type of g states that, if x and y are not aliased, then the final contents of the reference y are the same. Thus, the expressiveness of RF*, combined with its ability to use Z3 to discharge proof obligations, enables for the first time automated reasoning in the style of a relational Hoare logic, for proving noninterference properties of higher-order stateful programs.

2.3 Sampling, chosen-plaintext security, and one-time pads

We illustrate probabilistic reasoning using symmetric encryption schemes. Our goal is to communicate plaintexts between a sender and a receiver without leaking any information about their content. For simplicity, we assume messages with a fixed size n (called blocks, or bytes n) and do not involve active attackers. Padding and authentication can be easily added, but would complicate our presentation. We assume that the sender and the receiver share a secret key k (also a block), sampled uniformly at random by calling sample. We model this assumption by writing a single program where both parties are within the scope of this key. The simplest secure encryption scheme is the one-time pad, implemented for instance using bitwise XOR: to encrypt p, compute c = k ⊕ p; to decrypt c, compute p = c ⊕ k.

```plaintext
type block = b bytes { Length b = n }
let encrypt k p = xor k p
let decrypt k c = xor c k
```

In cryptography, confidentiality is usually stated as resistance against chosen-plaintext attacks (CPA) and encoded as a decisional game, in which an adversary chooses two plaintexts, receives the encryption of one of them under a fresh key, and must guess which of the two plaintexts was encrypted. (Decryption plays no role in this simple game; still, we may typecheck that it undoes encryption using classical refinements and properties of XOR.) This game may be coded in RF* as follows:

```plaintext
let cpa b p0 p1 = let p = if b then p0 else p1 in encrypt (sample n) p
where b is private and p0, p1, and the result are public. We thus express (perfect) CPA security relationally with the following type:
```

```plaintext
val cpa : b bool → eq block → eq block → eq block
```

In fact, viewing CPA security from an information flow perspective, a simpler formulation is possible. Instead of reasoning about two messages selected by b, we just need to show that the function:

```plaintext
let cpa p = encrypt (sample n) p has the type block → eq block.
```

This is the best type we can hope for encryption, treating the plaintext as private and the ciphertext as public. This more compact typing property subsumes the first one.

To prove secrecy for the one-time-pad, some probabilistic reasoning is called for. Indeed, operationally, calling sample n twice does not usually return the same value. However, from our formal development, we show that it is permissible to give
sample \( n \) a more specific relational type that allows us to complete the proof. In particular, we can type the call to \( \text{sample} \ n \) in a way that depends on the plaintext \( p \) and give it the type \( \text{mb:block}([\text{xor}(L\ p)\ (L\ m)=\text{xor}(R\ p)\ (R\ m)]) \). From this type, \( \text{RF}^* \) automatically proves \( \text{cpa}: \text{block} \to \text{eq}\ \text{block} \). Intuitively, this relational refinement is sound inasmuch as the distribution of the ciphertext is independent of the plaintext. This property is not specific to XOR; indeed, the only relational property that we require of two calls to \( \text{sample} \ n \) is that it return a pair of values related by a one-to-one function. As such, relational refinements in \( \text{RF}^* \) capture equivalences on the \textit{distribution of values} computed by a probabilistic program, rather than on the specific values themselves.

To this end, our library provides a polymorphic, typed variant of \( \text{sample} \) that takes as additional ghost parameter \( F \), a binary predicate on sampled values (of kind \( A \Rightarrow \text{A} \Rightarrow E \)), whose refinement states that it must be an injective function (or, equivalently, a bijection). Its type declaration in \( \text{RF}^* \) standard library is

\[
\text{type Function } F \equiv Y a . F a b \land Y a b , b a F a b 1 \to b = b 2 \\
\text{type Injective } F \equiv Y a b . F a b 1 \land F a b 2 \land b a 1 \to b a 2 \\
\text{val sample: } Y F . \text{len:}\text{nat}\{F (L b) (R b)\} 
\]

In our example, when calling \( \text{sample} \ n \) in \( \text{cpa} \), we instantiate \( F \) to the predicate \( \lambda \ b 0 . \ b 1 . \text{xor}(L\ b)\ b 0 = \text{xor}(R\ b)\ b 1 \), which is injective. Section 5 describes more realistic CPA schemes based on \( \text{sample} \).

### 2.4 Implicit flows and passport linkability

Before justifying our typing rules, notably for \( \text{sample} \), we present a concrete linkability attack against RFID-equipped passports, recently uncovered by Chothia and Smirnov [12]. We refer to their work for a detailed discussion. This attack is representative of common weaknesses in cryptographic implementations due to implicit flows in the handling of errors while processing decrypted data.

Following the ICAO specification for machine-readable travel documents, all recent European passports embed RFID tags featuring the Basic Access Control protocol, outlined in Fig. 1. The protocol has two roles, a passport tag and a reader, exchanging messages using short-distance wireless communications. The goal of the protocol is to establish a shared session key for accessing biometric data on the passport. The tag has a fixed key \( k \); the reader obtains \( k \) by scanning the passport—this step requires physical access to the passport, as it is presented to the reader.

The passport first samples a 64-bit nonce \( nt \) and sends it as a challenge to the reader. The reader samples its own nonce \( nr \) and some keying materials \( kr \), then encrypts the concatenation of these three values using \( k \). (Concretely, the protocol implements authenticated encryption as triple-DES-encryption concatenated with a plaintext MAC.) The passport decrypts, recomputes and checks the MAC to ensure that the message has not been tampered with, then compares the received nonce \( nt \) with the challenge, to confirm that the reader responded correctly. If both checks succeed, it generates its own keying materials \( kr \), appends it to the concatenation of the two nonces (in a different order than before), and computes the session key \( s = k t \oplus k r \). The reader then similarly decrypts the received ciphertext, checks the MAC, and computes \( s \).

We give below the code the tag uses for handling the encrypted message of the reader: \( \text{encrypt} \) and \( \text{decrypt} \) provide authenticated encryption; \( \text{concat} \) and \( \text{split} \) convert between triples of 64-bit values and their concatenation.

\[
\text{let tag1 k n t c = match decrypt k c with} \\
| \text{Some p \to let (nt',nr,kr) = split p in} \\
\quad \text{if nt = nt' then encrypt k (concat nr nt (sample_k t))} \quad \text{else nonceError} \\
| \text{None \to decryptError} 
\]

The code either produces an encrypted message, or it returns an error code. As written, it enables the following linkability attack:

1. The attacker eavesdrops any run of the protocol between a target passport and a honest reader, and records their second message.
2. Later, to test the local presence of this passport, the attacker runs the protocol (as the reader), replays the recorded message, and observes the response: although the protocol always fails to establish a key, the tag returns a \text{nonceError} if the two passports are the same, and a \text{decryptError} otherwise.

French passports reliably return different error messages, whereas other European passports return the same error message, but with measurably different timings. We interpret the attack as an implicit flow of information from the key used to decrypt to the error message. Indeed, if we type our keys as high confidentiality and the nonces and ciphers with \text{eq} refinements (inasmuch as they are exchanged on a public network), relational typechecking fails on the body of \text{tag1}. The result of the decryption is (a priori) not the same on both sides, so the cross-cases involve proving, for instance that when decryption returns \text{Some} \( p \) on the left and \text{None} on the right, the two resulting expressions are equal, which fails on the proof obligation \text{nonceError} = \text{decryptError}.

By ensuring that the same error messages are returned in both cases (i.e., by requiring that \text{nonceError} = \text{decryptError} this case is prevented. However, this alone is not sufficient for verifying the code. Naïvely, the cross-cases that arise when verifying the nested conditionals require proving (under a suitable relational path condition) that the encryption on the third line is indistinguishable from the error messages—which is patently false. However, by reflecting several cryptographic assumptions into detailed typing invariants in the protocol implementation, we can prove that such problematic cross-cases never actually arise (i.e., the path conditions guarding these cases are infeasible), and we can verify that this code preserves unlinkability. Specifically, we assume that the encryption is CPA, key-hiding, and CTXT (all specified by typing) and that there are no nonce collisions (the probability of a collision is less than \( q^2 2^{-64} \) where \( q \) is the number of sessions observed by the adversary). We show below the type we assign to \text{tag1}.

\[
\text{val tag1: k:}\text{key} \to \text{nt:}\text{nonce} \to \text{c:cipher} \to \text{irST} \text{pre block post} \\
\text{where pre h0 h1 = L nts R nts L c = R c} \land (L n t . k) \in \text{Sel h0 (L nts)} \land (R n t . k) \in \text{Sel h1 (R nts)} \\
\text{and post h0 h1 b0 b1 h0' h1' = b0=b1} 
\]

The pre-condition of \text{tag1} requires the nonces and ciphertext arguments to be equal (they are sent in the clear). Conversely, the keys may a priori differ. In addition, we maintain ghost state in a reference cell nts that holds a table modeling the association of nonces to keys—the last two clauses of the pre-condition require the nonce/key pairs to be present in this table. As a post-condition, \text{tag1} ensures that the returned blocks, b0 and b1, are equal. In the type, \text{irST} abbreviates the \text{RST} monad shown earlier, augmented with a heap invariant which places various constraints on the structure
of the nonce table \( \text{nts} \) and other mutable locations used in the implementation of this program. Arapinis et al. [2] also analyze this protocol, using the applied \( \pi \)-calculus, essentially proving unlinkability in a more abstract, symbolic model of cryptography.

3. Formal development

We formalize a core of \( r \mathsf{F} \) in the Coq proof assistant by developing \( \lambda_0 \), a minimal higher-order language with references, probabilistic assignments, and unbounded recursion. The formalization is based on the \( \mathsf{SSReflect} \) extension [18], and on the \( \mathsf{ALEA} \) library for distributions [3]. Overall, the formalization comprises over 5,000 lines of code excluding the aforementioned libraries.

The formalization is built in two steps. First, we consider a simply typed system \( G \vdash e : T \) for \( \lambda_0 \). Simple types \( T \) are extended to relational refinement types \( \mathcal{C} \) where one can add relational pre- and post-conditions to function types. This allows us to define a relational type system \( \mathcal{G} \vdash e_0 \sim e_1 : \mathcal{C} \) that relates a pair of expressions \( e_0, e_1 \) in the type \( \mathcal{C} \) under the relational context \( \mathcal{G} \).

We then give a denotational semantics for the introduced type systems. Simple types are given a set-theoretical interpretation \( \llbracket T \rrbracket \) in the standard way. Judgments \( \mathcal{G} \vdash e : T \) are interpreted as the elements of the form \( \{ e \}_I \), where \( I \) is any valuation for the context \( \mathcal{G} \). Taking into account that \( \lambda_0 \) is a language with references and probabilistic assignments, the denotation \( \{ e \}_I \) of \( e \) is defined as a function from memories (equivalently, states or heaps) to distribution over pairs composed of a memory and an element of \( \llbracket T \rrbracket \); we denote by \( M(\llbracket T \rrbracket) \) this function space. Relational types \( \mathcal{C} \) are interpreted as a relation \( \llbracket \mathcal{C} \rrbracket \) over \( M(\llbracket \mathcal{T} \rrbracket) \), where \( \mathcal{T} \) is the simple type derived from \( \mathcal{C} \) by erasing all refinements. This allows us to interpret a valid judgment \( \mathcal{G} \vdash e_0 \sim e_1 : \mathcal{C} \) by all the pairs of the form \( \{ \{ e_0 \}_I, \{ e_1 \}_I \} \in \llbracket \mathcal{C} \rrbracket \) for any pair of valuations \( (I_L, I_R) \) for the erasure of \( \mathcal{G} \).

3.1 \( \lambda_0 \): syntax

\( \lambda_0 \) is a simply typed \( \lambda \)-calculus with references and probabilistic assignments. For simplicity, we only consider two forms of probabilistic assignments: assigning a uniformly sampled boolean to a boolean variable (flip), and assigning an integer value sampled uniformly in a non-empty interval \([i, j] \) to an integer variable (pick\(_i^j\)). Formally, the sets of types, contexts, values and expressions are given by the following grammars:

\[
\begin{align*}
type & \ ::= \ B \mid \mathcal{T} \rightarrow \mathcal{T} \\
ctx & \ ::= \ \llbracket G \rrbracket, \{ x : T \} \\
value & \ ::= \ \llbracket e \rrbracket \mid \{ e \}_I \mid \text{fun } x : T \rightarrow e \mid \text{letrec } f \mid \text{let } x = e_1 \text{ in } e_2 \\
expr & \ ::= \ v \mid e \mid v \Rightarrow v \mid \text{flip}(v) \mid \text{pick}(v) \mid \text{let } x = e_1 \text{ in } e_2 \\
\end{align*}
\]

where \( x \) ranges over a set of variables, \( v \) ranges over a set of references and \( o \) ranges over a set of boolean operators, whose signature is of the form \( B \times \ldots \times B \rightarrow B \times \ldots \times B \). We assume that \( B \) contains the unit type (\( \mathsf{unit} \)) along with the types of booleans (\( \mathsf{bool} \)) and integers (\( \mathsf{int} \)). Their associated constructors are \( \bullet, \mathsf{true}, \mathsf{false} \) and \( n \) for \( n \in \mathbb{N} \). We implicitly assume that each reference has an ambient base type, and write \( r : \mathsf{ref} \mathcal{T} \) to denote that \( r \) is a reference of type \( \mathcal{T} \). The dynamic semantics is defined in the standard way as the compatible closure (for a call-by-value convention) of the \( \beta \mu \delta \)-contraction.

3.2 \( \lambda_0 \): typing

As usual, a typing context is a sequence of bindings \( x : T \) such that a variable is not bound twice. The typing rules for deriving valid judgments \( \mathcal{G} \vdash v : T \) and \( \mathcal{G} \vdash e : T \), in a simply typed setting, are standard and omitted.

3.2.1 Relational refinement types

Relational assertions are formulae over tagged variables \( x_C \) or \( x_R \) and tagged references \( r_C \) or \( r_R \); informally, tags determine whether the interpretation of \( x \) or \( r \) will be taken w.r.t. the left or right projection of a relational valuation. In order to interface with automated first-order provers, relational assertions are first-order \( \mathcal{B} \)-sorted formulae built from operators in \( \mathcal{O} \) and predicates taken from a set of \( \mathcal{B} \)-sorted predicates which includes at least the equality predicates for all the base types. Note that tagged variables always occur free in assertions, and only logical variables can be bound. The formal definition of well-formed assertion is as expected, and omitted; we write \( \mathcal{G} \vdash \Phi : \mathsf{Prop} \) if \( \Phi \) is a well-formed assertion under \( \mathcal{G} \). For instance, the relational assertion \( \forall y : \mathsf{int}. x_C \leq y \Rightarrow x_R \leq y + r_R \) is well-formed under any context \( \mathcal{G} \) s.t. \( x : \mathsf{int} \in \mathcal{G} \), assuming that \( r : \mathsf{ref} \mathsf{int} \).

Refinement types are either relational types (denoted by \( \mathcal{T}, \mathcal{U} \)), which will be used for relational typing of values, or computation types (denoted by \( \mathcal{C} \)), used for relational typing of expressions. They are defined by the following grammar:

\[
\begin{align*}
\mathcal{T} & \ ::= \ B \mid \{ x : T \} \rightarrow \mathcal{T} \\
\mathcal{C} & \ ::= \ \{ \Phi \} y : \mathcal{T}(\Psi) \\
\end{align*}
\]

where \( \Phi \) and \( \Psi \) are relational assertions. By convention, \( x \) and \( y \) are bound in \( \{ x : B \} \rightarrow \mathcal{C} \) and \( \{ \Phi \} y : \mathcal{T}(\Psi) \), respectively. However, the type system enforces that \( x \) and \( y \) only occur in \( \mathcal{C} \) and \( \Psi \) respectively, if \( \mathcal{T} \) is a base type. In other cases, we write \( \mathcal{T} \rightarrow \mathcal{C} \) and \( \{ \Phi \} \mathcal{T}(\Psi) \).

A relational context \( \mathcal{G} \) is a sequence of bindings \( x : T \) s.t. a variable is not bound twice. The refinement type \( \mathcal{C} \) is a valid refinement of \( \mathcal{T} \) under \( \mathcal{G} \), written \( \mathcal{G} \vdash \mathcal{C} \triangleleft \mathcal{T} \), if \( \mathcal{T} \) is the result of erasing all pre- and post-conditions occurring in \( \mathcal{C} \) and if any assertion that appears in \( \mathcal{C} \) is well-formed in \( \mathcal{G} \) augmented by the local context of \( \Phi \) in \( \mathcal{C} \). This relation is lifted to relational contexts: \( \mathcal{G} \vdash \mathcal{C} \) is the smallest relation s.t. \( \triangleleft \mathcal{G} \) and if \( \mathcal{G} \vdash \mathcal{C} \triangleright \mathcal{T} \) then \( \mathcal{G} \vdash \mathcal{C} \rightarrow \mathcal{T} \), \( x : T \rightarrow \mathcal{C} \).

3.2.2 Relational typing

Figure 2 gives a significant subset of the rules defining the relational typing judgments \( \mathcal{G} \vdash v_1 \sim v_2 : \mathcal{T} \) and \( \mathcal{G} \vdash e_1 \sim e_2 : \mathcal{C} \) for respectively values and expressions. The full set of rules appear in the accompanying Coq formalization.

A judgment of the form \( \mathcal{G} \vdash e_1 \sim e_2 : \{ \Phi \} \mathcal{T}(\Psi) \) is valid when for any pair of initial memories \( m_1, m_2 \) satisfying the pre-condition \( \Phi \), the pair of distributions over memories obtained by executing \( e_1 \) and \( e_2 \) are related by the post-condition \( \Psi \) (more exactly, by the lifting of \( \Psi \) to distributions, defined below). Intuitively, when \( \Psi \) is an equivalence relation, observing to which equivalence class the results belong does not help in distinguishing the two expressions.

The rules come in two flavors: double- or single-sided. Double-sided rules allow relating programs with the same head symbol. For instance, rules \( \mathsf{Leq} \) and \( \mathsf{App-Base} \) are double-sided. They work by relating sub-expressions pairwise, composing pre- and post-conditions using the implicit order of evaluation. Rule \( \mathsf{Leq} \) emphasizes this, where the post-condition of the let-bound expression is the pre-condition of the body.

It is not always possible to progress using double-sided rules. For instance, one may want to show that the two expressions (if \( b \) then \( v \) else \( v \)) and \( v \) are related by a suitable post-condition. The two expressions having different head symbol, no double-sided can apply. Single-sided rules allow to overcome this limitation. The rule \( \mathsf{It} \), which permits to relate an if-expression to an arbitrary expression, is an example of a single-sided rule.

Rules for reference assignment (\( \mathsf{Ref} \) and \( \mathsf{Ref-Left} \)), which come in two flavors too, make use of the ability to write assertions about the resulting memory. For example, the two expressions \( r := b \circ \). 2012/12/4
and $r : b_1$ are related by a post-condition $\Psi$ when $\Psi$, where all occurrences of $r \tau$ (resp. $r \kappa$) have been replaced by $b_0$ (resp. $b_1$), holds as a pre-condition.

Up to now, we only considered rules for expressions headed by non-probabilistic constructions. Rules for random sampling are double-sided and require the existence of a bijection $f$ between the range of the two distributions, ensuring a one-to-one correspondence between related values. In the case of flip, we explicitly give the only 2 existing bijections from $\text{bool}$ to $\text{bool}$.

### 3.3 $\lambda_\mu$: denotational semantics

#### 3.3.1 Background

The denotational semantics of well-typed expressions is based on the sub-distribution monad [3, 25]. For every set $X$ and $x \in X$, let $\delta_x : X \to [0, 1]$ be the Dirac function for $x$: i.e., $\delta_x(x) = 1$ and $\delta_x(y) = 0$ if $x \neq y$. A sub-distribution over $X$ is a function $\mu : (X \times [0, 1]) \to [0, 1]$ such that for every $f : X \to [0, 1]$, $\mu f = \sum_{x \in X} (\mu \delta_x)(f(x)) \leq 1$. A distribution is a sub-distribution $\mu$ such that $\mu(\lambda x.1) = 1$. The range $\mu$ of a sub-distribution $\mu$ is the smallest set $X_0 \subseteq X$ containing all $x \in X$ such that $\mu \delta_x \neq 0$. A sub-distribution $\mu$ is discrete iff its range is a discrete set; note that every finite set $X$ induces a discrete distribution $\mathbb{I}_X$, that assigns probability $\frac{1}{|x|}$ to each element of $X$.

Let $D(X)$ be the set of discrete sub-distributions over $X$. $D(X)$ inherits from $[0, 1]$ the structure of a complete lattice. Moreover, sub-distributions can be given the structure of a monad, by taking as unit and composition operators:

- unit : $X \to D(X)$
- bind : $D(X) \to (X \to D(Y)) \to D(Y)$

The interpretation of relational types rests on an operator $^r$ that lifts relations over $A \times B$ into relations over $D(A) \times D(B)$; the operator is inspired from early works on probabilistic bisimulations [19], and used in EasyCrypt to interpret relational judgments. Formally, let $\mu_1 \in D(A)$ and $\mu_2 \in D(B)$; then $^P \mu_1, \mu_2$ iff:

$\exists \mu : D(A \times B), \pi_1(\mu) = \mu_1 \land \pi_2(\mu) = \mu_2 \land \text{range } \mu \leq P$

where $\pi_1$ and $\pi_2$ are the projections for distributions over pairs.

#### 3.3.2 Set-theoretical interpretation

We assume each base type $B$ is given a set-theoretical interpretation $[B]$, and that each constructor $c$ belonging to the base type $B$ is given a denotation $B(c) \in [B]$. We define the set of semantical values as $\bigsqcup [B]$, and then the set $M$ of states as the set of well-typed mappings from references to semantical values. Then we extend the interpretation to functional types by setting $[T_1 \to T_2] = \bigsqcup [T_1] \to M([T_2])$

where $M(X) \cong M \to D(M \times X)$.

A valuation $I$ is a function that maps every declaration $x : T$ to a semantical value. A valuation $I$ is valid for $G$, written $I \in G$, if $I$ maps every declaration $x : T$ in $G$ to an element of $[T]$. Let $I$ be a valuation and $m$ be a memory. The interpretations $[v]_I$ of a value $v$ and $[e]_I$ of an expression $e$ are defined in Figure 3. If $I$ is a valid valuation for $G$, and $G \vdash_v v : T$ is derivable, then $[v]_I \in [T]$. Likewise, if $G \vdash_x e : T$ is derivable then $\text{eval}_m([e]_I) \in M([T])$.

#### 3.3.3 Relational interpretation

A relational valuation $I$ for $G$, written $I \models G$, is a pair of valuations for $G$. If $I = (I_L, I_R)$, we write $\pi_1(I)$ (resp. $\pi_2(I)$)
for \( I_C \) (resp. \( I_R \)), and \( I(x) \) for \((I_C(x), I_R(x))\). We assume given a relational interpretation for formulas, written \( [\Phi]_x \), s.t. for any formula \( \Phi \) well-formed under \( G \), for any relational valuation \( I \models F \), \( [\Phi]_x \) is a relation on \( M \). This relation is defined as usual, using the left/right valuation (resp. left/right memory argument) for interpreting variables on the left/right.

Figure 4 defines the interpretation of valid relational types, written \( \langle G \models T \triangleleft T \rangle_x \), and valid computation types, written \( \langle G \models T \triangleleft T \gamma \rangle_x \) w.r.t. a relational valuation \( I \). A relational valuation \( I \) is valid w.r.t. \( G \), written \( I \models G \), if \( G \preceq G \), and for every variable \( x \) declared in \( G, I(x) \) \in \( \langle G \models G(x) \triangleleft G(x) \rangle_x \).

Finally, we define the semantic validity of judgments: we say that two values \( u_1 \) and \( u_2 \) are semantically related in \( T \) under \( G \), written \( G \models u_1 \sim u_2 : T \), if \( G \preceq G \), and \( G \models T \triangleleft T \).

\[
\forall I \models G. \langle \{u_1\}_{\pi_1(x)}(x), \{u_2\}_{\pi_2(x)}(x) \rangle \in \langle G \models T \triangleleft T \rangle_x
\]

We say that two expressions \( e_1 \) and \( e_2 \) are semantically related in \( C \) under \( G \), written \( G \models e_1 \sim e_2 : C \), if \( G \preceq G \), and \( G \models T \triangleleft T \), and

\[
\forall I \models G. \lambda. [\pi_1(e_1(x)) \triangleleft \pi_2(e_1(x)) \triangleleft \pi_2(e_2(x))] \in \langle G \models C \triangleleft C \rangle_x
\]

The following theorem states that all judgments of the logic are sound w.r.t. their interpretation.

**Theorem 1** (Soundness).

- If \( G \models v_1 \sim v_2 : T \), then \( G \models v_1 \sim v_2 : T \).
- If \( G \models e_1 \sim e_2 : C \), then \( G \models e_1 \sim e_2 : C \).

Technically, we prove the soundness of each rule as a lemma, directly from the semantics. It allows one to fall back on the full generality of the Coq system whenever reasoning outside of the logic is required.

4. **Encoding \( \lambda p \) in \( R^* \)**

There are two key ideas behind our encoding of \( \lambda p \) in \( R^* \). First, as shown in §2.3, we introduce probabilistic computations into \( R^* \) axiomatically, by providing a sample primitive at the appropriate type. Progammers can instantiate \( sample \) at runtime by providing a suitable source of randomness. Next, as discussed in §2.2, we adapt the Hoare state monad \( ST \) to a monad \( RDST \) for computations with relational pre- and post-conditions. We provide here more details about our encodings, in particular the style we adopt to compute relational VCs for the \( RST \) monad, and the manner in which we reuse classical specifications.

4.1 **Representing \( \lambda p \) types**

To implement \( \lambda p \), we begin with a translation of its types into \( F^* \) augmented with a relational state monad. To stay close to \( \lambda p \), our translation uses a monad \( RST \), which we then adapt to the monad \( RDST \) of §2. Like in \( \lambda p \), post-conditions in \( RST \) only relate the output data and heaps, not the initial heaps. Specifically, the type \( RST \) \( \text{pre a post } \) can be interpreted in \( R^* \) as a store-passing function (or a primitive \( \text{heap} \) with the signature shown below:

\[
\text{RST pre a post } = h: \text{heap}(\text{pre (L h) (R h)}) \rightarrow (\text{x:a} \rightarrow \text{h':heap} \text{ post (L x) (R x) (L h') (R h')})
\]

The type translation is homomorphic on most of \( \lambda p \)'s typing constructs, with the interesting cases mainly on the computation types. For first-order computation types \([\Phi]_x \) \( \mathcal{B} [\Psi]_y \) is \( \mathcal{RST} [\Phi] \mathcal{B} [\Psi] \), while, higher-order computation types in \( \lambda p \), \([\Phi]_x \mathcal{L} [\Psi]_y \) (whose post-condition is not dependent on the result \( \ell \)), are represented as \( \mathcal{RST} \mathcal{L} [\Phi] \mathcal{L} [\Psi] \).

\[1 \text{ In principle, in } R^*, \text{ one could write types like } RST \ p \rightarrow \ RST \ \ p \times T \ q' (\lambda f_0 f_1 q) \text{ where } q \text{ mentions } f_0, f_1 \text{ which is inexpressible}\]

4.2 **A monad of predicate transformers for \( VC \) generation.**

Next, to provide type inference for \( R^* \), rather than writing relational Hoare triples in \( RST \), we write specifications using predicate transformers. This style is adapted from the Dijkstra state monad, previously introduced for inferring classical (non-relational) verification conditions for stateful \( F \) programs [28]. In particular, we introduce the relational Dijkstra state monad, \( RDST \), and show its signature below. (Note, we write polymorphic types implicitly assuming their free type variables are prefix quantified.)

- **type RDST a wp = \text{wp}. RST0 (wp p) a p**
- **val return x:a → RDST a (\text{Ap} p (L x) (R x))**
- **val bind: RDST a (wp1 \rightarrow (x:a → RDST b (wp2 x))) → RDST b (Ap:λh0 h1. wp1 (λx0 x1 h0' h1')) ((\forall \ x. L x=00 \land R x=11 \Rightarrow wp2 x p h0' h1')) h0 h1**

The type \( RDST t \ wp \) is an abbreviation for the \( RST0 \) monad that is polymorphic in its post-condition. Specifically, \( RDST t \ wp \) is the type of computation which for any relational post-condition \( p \) on ts and heaps, the pre-condition on the input heaps is given by \( wp \).

Unlike the Hoare-style \( RST0 \) monad, the \( RDST \) monad yields a weakest pre-condition calculus by construction. As indicated by the signature of \( bind \), when composing computations in the \( RDST \) monad, we simply compute a pre-condition for the computation by composing the predicate transformers of each component. A slight complication arises from the need to constrain the formal parameter \( x:a \) of \( wp2 \) relationally. In general, \( wp2 \) will have free occurrences of \( L x \) and \( R x \). We relate these to the result of the first computation using the guard \( L x=x00 \land R x=11 \), before composing \( wp1 \) and \( wp2 \).

Additionally, by exploiting the post-condition parametricity of \( RDST \), we can recover the expressiveness of a 6-place post-condition relation in the \( RST \) monad that we use in our examples. We show the definition of \( RST \) below.

- **type RST pre a post = RDST a (Ap:λh0 h1. pre h0 h1 \land ∀x0 x1 h0' h1'. post h0 h1 x0 x1 h0' h1' \Rightarrow p x0 x1 h0' h1')**

4.3 **Lifting classical specifications.**

To promote reuse of existing verified \( F^* \) code in \( R^* \), we provide combinators to lift specifications written with classical predicate transformers into the \( RDST \) monad. To illustrate, we show the \( R^* \) specifications of primitive operations on references—the same combinators apply to arbitrary classically verified code.

- **type lift wp0 wp1 p h0 h1 = wp0 (λx0 h0' . wp1 (λx1 h1'. p x0 x1 h0' h1')) h1 h0**
- **type Rd x p h = p (\text{Sel} h x) h**
- **val (!) : x:ref a → RDST a (\text{lift (Rd (L x)) (Rd (R x)))**
- **type W x v p h = p () (U pd h x v)**
- **val (=): x:ref a → v:ref a → RDST unit (\text{lift (W (L x)) (L v)) (W (R x) (R v))**

The combinator \( \text{lift} \) takes two classical predicate transformers \( wp1 \) and \( wp0 \) and composes them by, in effect, “running” them separately on the heaps \( h0 \) and \( h1 \) and relating the results and heaps using the relational post-condition \( p \). The types given to \( \text{deref} \) and assignment should be evident—these are simply the relational liftings of the standard, classical weakest pre-condition rules for these constructs (\( Rd \) and \( W \), respectively).

4.4 **Computing relational VCs.**

We re-purpose the bulk of typechecking of \( F^* \) to \( R^* \). Although the relational typing rules of Fig. 2 generally analyze a pair of programs \( e_0 \sim e_1 \), for the most part, we are concerned with proving

in \( \lambda p \). However, such a type is generally useless in \( R^* \), since the functions \( f0, f1 \) cannot be applied in \( a \).
relational properties of multiple executions of a single program. Thus, in the special symmetric case where we are analyzing $e \sim e$, the rules of Fig. 2 degenerate into the standard typing rules for monadic $F^\star$ (which is parametric in the choice of monad, so configuring it to use RDST is easy).

The main subtlety in computing relational VCs arises when analyzing the cross-cases of conditional expressions—for this we implement the single-sided rules in the judgment, and attempt to revert to the symmetric case as soon as we detect that the program fragments are indeed the same. For example, the rule [It] allows us to relate $b$ then $e$ else $e'$ if generating subgoals for $e \sim e$ and $e' \sim e$, where, at least the former can be handled once again by the symmetric rules.

The rule [Red] of Fig. 2 is impossible to implement in full generality—it permits reasoning about stateful programs after arbitrary reductions of open terms. However, the rule is approximated by the $F^\star$ typechecker for terms that can be given classical predicate transformer specifications. In particular, when trying to relate $e_0 \sim e_1$, if we can use the symmetric judgments and use $e_0 \sim e_0 : RDST t$ (lift wp0), and $e_1 \sim e_1 : RDST t$ (lift wp1), then we type $e_0 \sim e_1$ at type $RDST t$ (lift wp0 wp1). In effect, by making use of classical predicate transformers on either side, we approximate the reduction relation for stateful terms used by [Red] (and its symmetric counterpart).

All these measures for handling the asymmetric cases are still incomplete. When trying to prove a relation between $f v_0 \sim g v_1$ in a context $G$ with relational types for $f$ and $g$ that cannot be decomposed into a pair of classical specifications, it becomes impossible to complete the derivation. In such cases, $F^\star$ emits \textbf{False} as the VC (guarded by a relational path condition). Nevertheless, it may still be possible to discharge the VC, if the path condition is infeasible. This is the case, for example, when trying to relate the result of \textbf{encrypt} with \textbf{errorNonce} in the passport example of §2.4.

4.5 Proving VCs using Z3.

Once a VC has been computed, we ride on an existing encoding of VCs computed for the classic Dijsktra monad within Z3. We rely on a theorem from Schlesinger and Swamy [28] which guarantees that despite the use of higher-order logic when computing VCs, once a predicate transformer is applied to a specific first-order post-condition, so long as there is no inherent use of higher-order axioms in the context, a first-order normal form for the VC can be computed.

5. Applications.

Table 1 summarizes our experimental evaluation of $F^\star$. For each program, we give the number of lines of code and type annotations (excluding comments), and the typechecking time, which is mostly dominated by the time spent solving VCs in Z3. For lack of space, most of these examples are only briefly described, with a more detailed discussion of the last two programs, \textbf{counter} (a cryptographic construction) and \textbf{meter} (a privacy protocol).

Information flow and passport The first five programs provide many information flow examples and test cases for single-sided rules using several variations of the RDST monad construction of §4. The sixth program, passport, was discussed in §2; its verification involves modelling key-hiding symmetric encryption.

Up to bad The program \textbf{uptobad} illustrates a common cryptographic proof pattern to prove refinement formulas of the form $\varphi \lor$ Bad where $\varphi$ is the property we are interested in and Bad captures conditions that may cause the program to ‘fail’, usually with a small probability. To avoid polluting all our specifications with this disjunction, we define an up-to-bad variant of the RDST monad, where all pre- and post-conditions, and heap invariants are only enforced as long as a distinguished location, bad: \textbf{ref bool}, is false. Intuitively, this adds an implicit $\lor$ Bad to every refinement. Independently, we can compute (or bound) the probability of Bad being set to true; for passports for instance, we set bad to true when we detect a collision between two sampled nonces nt, and bound its probability with $q^2/2^q$ where $q$ is the number of sessions.

Random Oracle The program \textbf{ro} provides an idealized implementation of a cryptographic hash function in the random oracle model. The implementation uses a mutable reference holding a table map of a cryptographic hash function in the random oracle model. Random Oracle Querying the table, including proving that the tables grow monotonically; that in every pair of executions, the tables agree on the fragments corresponding to queries made by both honest participants and adversaries. To verify the program, we carry several invariants on this table, including proving that the tables grow monotonically; that in every pair of executions, the tables agree on the fragments corresponding to queries made by A; and that on the fragments corresponding to queries by H, the entries are related by an injective function that ensures they have indistinguishable distributions. The interface of \textbf{ro} is designed to allow the full use of relational sample on the H fragment, and to account for failure events (e.g., returning a value to A that collides with one that was already provided to H), allowing for its modular use in a context that must bound their probability.

CCA2 We program an ideal, stateful functionality for CCA2 public-key encryption that maintains a log of prior oracle encryptions, similar to those of Fournet et al. [16]. They require that all code that operates on secrets be placed in a separate module that exports plaintexts as an abstract type. Using relational types, we lift this restriction, enabling us to verify code that use encryption without restructuring. The code is essentially higher-order, simulating ML functors using a dependently typed record of functions.

Nonces and Private Authentication Exploiting the modularity of our CCA2 implementation, we verify \textbf{simplesnatch}, a protocol that implements a common authenticated pattern based on fresh nonces, formalizing the intuition “if $a$ encrypts a fresh random nonce using the public key of $b$, and later decrypts a response containing that nonce, then the whole response must have been sent by $b$”. We further extend \textbf{simplesnatch} to \textbf{privateauth}, a private authentication protocol that allows two parties to communicate securely, protecting both their messages and identities from third parties.

El Gamal encryption The chosen-plaintext security of ElGamal encryption is a classic example of a code-based cryptographic proof.
inf. flow with state and single-sided rules
inf. flow with arithmetic
up-to-failure reasoning
nonces with private authentication
variation of pure1
defense against attack on French passports
authentication with nonces
mode
counter mode for AES

can be typed as \( \langle \text{R} \rangle \text{p} \text{i} \cdot \text{ps} \text{RST} \text{pre} \text{block} \text{post} \)

\text{where} \quad pre \text{h}0 \text{h}1 = (\langle \text{Requires: i not in the log yet} \rangle)
not (\text{in} (\text{L}\text{i}) (\text{Domain} (\text{Sel} h0 (\text{L} \text{log}))))
\land (\text{L}\text{i} = \text{R}\text{i}) \land (\text{Seqn} (\text{L}\text{i}) < \text{Sel} h0 (\text{L} n))

\text{and post} \text{h}0 \text{h}1 \text{m}0 \text{m}1 \text{h}0' \text{h}1' = (\langle \text{Ensures: log extended with \langle i,p,m \rangle} \rangle)
\text{Mask} \text{(L}\text{p}) (\text{R}\text{p}) \text{m}0 \text{m}1 (\langle \text{and sampled \text{m} is related by injectivity} \rangle)
\wedge h0' = \text{Upd} h0 (\text{L} \text{log}) (\text{Entry} (\text{L}\text{i}) (\text{L}\text{p}) \text{m}0); \text{Sel} h0 (\text{L} \text{log})
\wedge h1' = \text{Upd} h1 (\text{R} \text{log}) (\text{Entry} (\text{R}\text{i}) (\text{R}\text{p}) \text{m}1); \text{Sel} h1 (\text{R} \text{log})

\text{let} f p i = \text{match assoc i !log with}
| \text{Some}_m (\text{m}) \rightarrow \text{m} (\langle \text{unreachable} \rangle)
| \text{None} \rightarrow \text{sample} \text{p}

When using a counter (function encrypt), typechecking relies on a joint invariant on the counter \( n \) and the content of the log, that states that all entries in the log have an index block \( i \) formatted from some \( n' < n \). It also involves excluding counter overflows, and assuming that toBytes is injective. This enables us to prove that our encryption is secure with no loss in the reduction: the advantage of a CPA adversary against our code is the same as the advantage of some adversary against the PRF assumption.

When using instead fresh random blocks (function encrypt'), the situation is more complex, as there is a non-null probability that two different encryptions pick the same index \( i \). Our construction is secure as long as no such collisions happen. We capture this event using the ‘up to bad’ approach presented above, for a Fresh module that silently detects collisions and sets the bad flag accordingly. Concretely, the probability of having a collision when sampling \( q \) blocks of 16 bytes each is bounded by \( q^2/2^{258} \).

By typing, we prove that encryption, and any program that may use it, leaks information only once \text{bad} is true. Thus, we prove the concrete security of our scheme with a loss of \( q^2/2^{258} \) in the reduction to PRF.

<table>
<thead>
<tr>
<th>Name</th>
<th>LOC</th>
<th>TCU</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>arith</td>
<td>43</td>
<td>4.3</td>
<td>inf. flow with arithmetic</td>
</tr>
<tr>
<td>pure1</td>
<td>35</td>
<td>1.7</td>
<td>inf. flow with inference</td>
</tr>
<tr>
<td>pure2</td>
<td>33</td>
<td>1.7</td>
<td>Variation of pure1</td>
</tr>
<tr>
<td>st</td>
<td>52</td>
<td>3.6</td>
<td>inf. flow with state</td>
</tr>
<tr>
<td>singlesided</td>
<td>111</td>
<td>14.2</td>
<td>inf. flow with state and single-sided rules</td>
</tr>
<tr>
<td>passport</td>
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<td>44.2</td>
<td>Defense against attack on French passports</td>
</tr>
<tr>
<td>uptobad</td>
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<tr>
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<td>88</td>
<td>6.5</td>
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<td>simplenonce</td>
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<td>ElGamal encryption</td>
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<tr>
<td>counter</td>
<td>106</td>
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<td>Counter mode for AES</td>
</tr>
<tr>
<td>meter</td>
<td>182</td>
<td>79.8</td>
<td>Pedersen commitments for smart meters</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>1378</strong></td>
<td><strong>451.3</strong></td>
<td>**</td>
</tr>
</tbody>
</table>
the privacy of the detailed readings. The protocol relies on Pedersen commitments [24] and public-key signatures. We prove perfect, information-theoretic privacy (entirely by relational typing) and computational integrity by reduction to the discrete log problem (using "up-to-bad").

**Homomorphic Pedersen commitments** We implement (and type) commitments, parameterized by some multiplicative group of prime order $p$. We outline their interface and review their main security properties.

type opening pp pp =
  c-opng { Eq ( (trap (L pp) + L o) o (trap (R pp) + R o)) }
val sample pp pp x =
  c-xtext opening pp pp x 
val commit pp pp x =
  c-text opening pp pp x 
let commit pp x r =
  c-mmpp (Eq pp x r)
let verify pp x r c =
  c-rec (Eq pp x r c)

Public parameters $pp$ consist of the prime $p$ and two different group generators $g$ and $h$ (possibly chosen by the utility). A commitment to $x$ with opening $o$ is a group element $c = g^x h^o$. Although assumed hard to compute, there exists $\alpha$ (known as the trapdoor for these parameters) such that $g = h^\alpha$. Accordingly, we use $\alpha = \text{trap} pp$ for specification purposes, in refinement formulas but not in the protocol code. We use $\alpha$ in particular to provide an injective function for randomly sampling the opening $o$ (modulo $p$) so that it perfectly hides $x$: the relational refinement type opening in the postcondition of sample records that $\alpha (L x) + (L o) = \alpha (R x) + (R o)$, which implies $g^{x'} h^{L o} = g^{x o} h^{R o}$ and enables us to type the result of commit as public (eq int). Intuitively, every commitment can be opened to any $x$, for some hard-to-compute $o$, so the commitment itself does not leak any information about $x$ as long as $o$ is randomly sampled and kept secret.

Commitments can be multiplied: $g^x h^o = g^{x'} h^{o'} \rightarrow g^{x+x'} h^{o+o'}$ and exponentiated: $(g^x h^o)^y = g^{x y} h^{o y}$ to compute linear combinations of their exponents without necessarily knowing them. These observations preserve eq and opening relational refinements.

**Smart Meter** We show some typed code for each role of the protocol. We have abstract predicates $\text{Readings}$ and $\text{Rates}$, and enables us to type the result of commit with opening.

val make_payment: pp pp x =
  c-rec { Eq pp x x r c }

The relational part of the pre-condition (enclosed with (\ldots)) is a declassification condition, capturing the user’s intent to leak the total payment, computed as the scalar product ($SP$) of the detailed readings and rates. By typing the code of the double scalar product sums, we get the same equation for the openings. The result type of make_payment tells us that those two scalars reveal no further information on any readings leading to the same fee.

**Utility** The utility verifies the signature on the commitments $cs$, and uses the rates $r$ to computes the product of exponentials

\[
\prod_i c_i^{p_i} = \prod_i (g^{x_i h^o})^{p_i} = \prod_i g^{x_i p_i} h^{o p_i} = g^{\sum_i x_i p_i} h^{\sum_i o p_i}
\]

and compares it to the commitment $g^x h^o$ computed from the values $x$ and $o$ presented by the user. Unless the user can open a commitment to several values $x$ (which can be further reduced to the discrete log problem), this confirms that $x$ is the correct payment. To type the verifier code, we write classic, but non-trivial refinements, using ghost scalar products to keep track of its computation.

### 6. Related work and conclusions

Our work spans protocol and relational program verification.

**Protocol verification** Blanchet’s recent account of the field of protocol verification provides a panorama of existing tools and of major achievements [9]. Most of the literature focuses on verifying protocol specifications, or protocol implementations through model extractors; however, our work is most closely related to approaches that reason directly about implementations in the symbolic [8, 15] or computational models [16, 20]. Alternatives include generating implementations from verified models [10], and extracting models from implementations [1].

**Relational program verification** Relational Hoare Logic was first introduced to reason about program optimizations and information flow in core imperative programs [7]. It was later extended to probabilistic programs, and used to reason about cryptography and differential privacy [4, 6]; and to higher-order programs with store (but not probabilities) and used to reason about advanced information flow policies using interactive proofs in Con [23]. Relational logics can also be used to reason about continuity [11]. Naturally, numerous program analyses and specialized relational logics enforce 2-properties of protocols.

**Conclusions** RPF is a full-fledged programming language that supports fine-grained relational reasoning about probabilistic programs, and mechanisms to exploit localized guarantees obtained by such means in global program analyses. We aim in the future to apply RPF to certify the end-to-end security of large protocol implementations, such as ongoing efforts with F* to verify TLS 1.2. Along another axis, we aim to use RPF to verify cryptographic implementations and advanced cryptographic constructions that are inherently higher-order and hence out of reach for existing relational tools, like EasyCrypt. Examples of such “higher-order cryptography” include leakage-resilience, which accounts for side channel attacks, and key-dependent message security.

### References
