Billiards and Ramified Covers of Curves

Joint with P. Hubert

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Abstract

Given a Euclidean polygon with vertex angles rational multiples of π, one can glue together copies of the polygon (and reflections through its various sides), to construct a surface. Straight-line paths with reflection off of the sides can then be studied as geodesics on the constructed flat surface. Moreover, the complex structure of the plane induces a complex structure on the surface; dz induces an abelian differential.

Conversely, integration of an abelian differential ω on an algebraic curve X induces a flat structure. Since $SL(2, R)$ acts on the plane, there is a natural action of this group on the space of abelian diffentials on algebraic curves of fixed genus. The orbit of any fixed pair (X, ω) projects to the Riemann moduli space of algebraic curves. On rare occasions, this project is itself an algebraic curve.

We report on joint work with Pascal Hubert on the projections into moduli space of the orbit of pairs

(X, ω) when ω is the pull-back of an abelian differential by way of a ramified covering.

I. Billiards to Surfaces: Torus

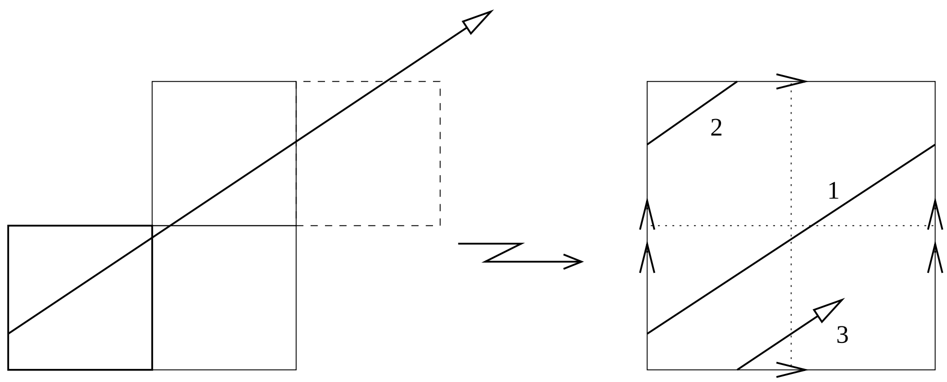
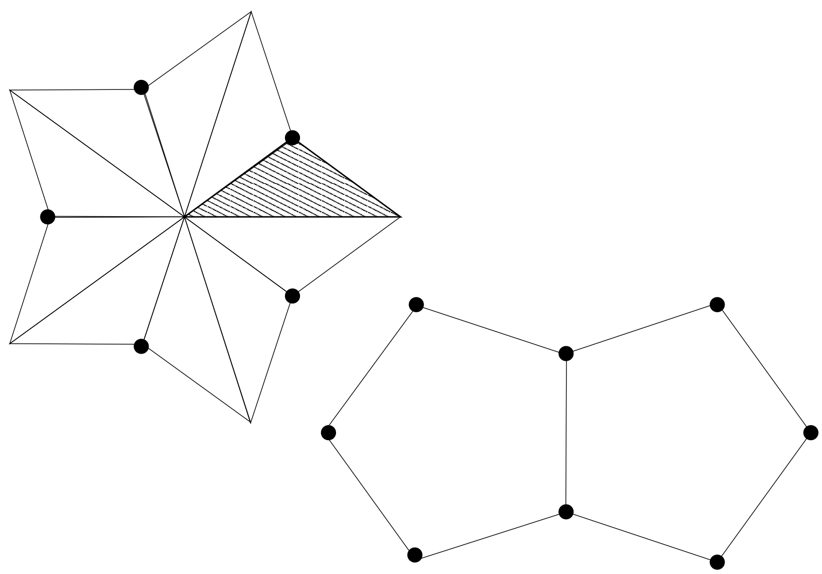


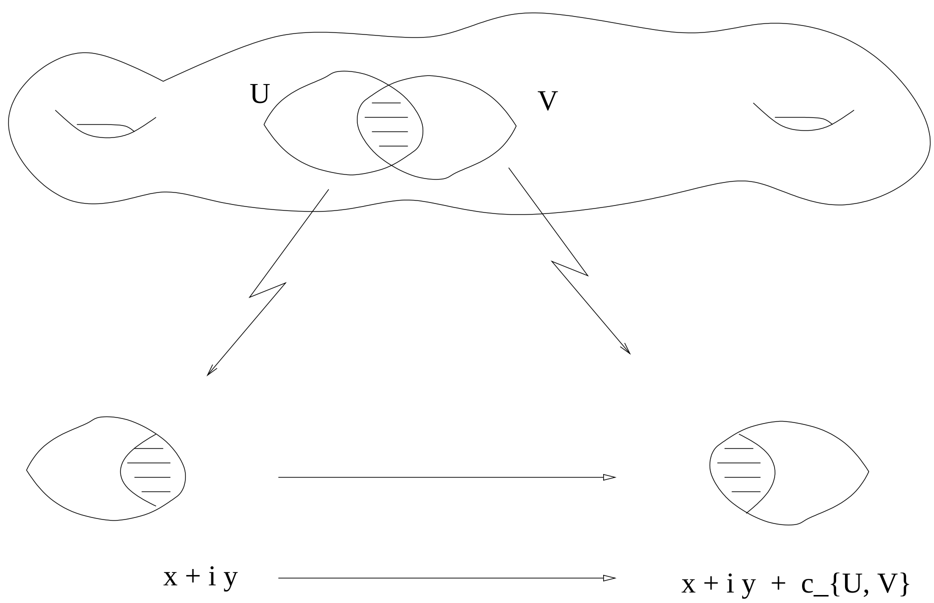
Figure 1: Unfolding; square table to torus surface.

Billiards to Surfaces: Genus 2

Triangle with angles $(\pi/5, \pi/5, 3\pi/5)$ yields a genus two surface: flat except for one point of angle 6π.



Translation Surfaces



The idea of translation surface.

Holomorphic 1-forms

Riemann surface X and holomorphic 1-form ω.

Locally $\omega = f(w)dw$.

Point $p_0 \in X$, gives local coordinates

$$z(p) = \int_{p_0}^{p} \omega \, .$$

In these local coordinates, $\omega = dz$.

1-form <—> translation structure

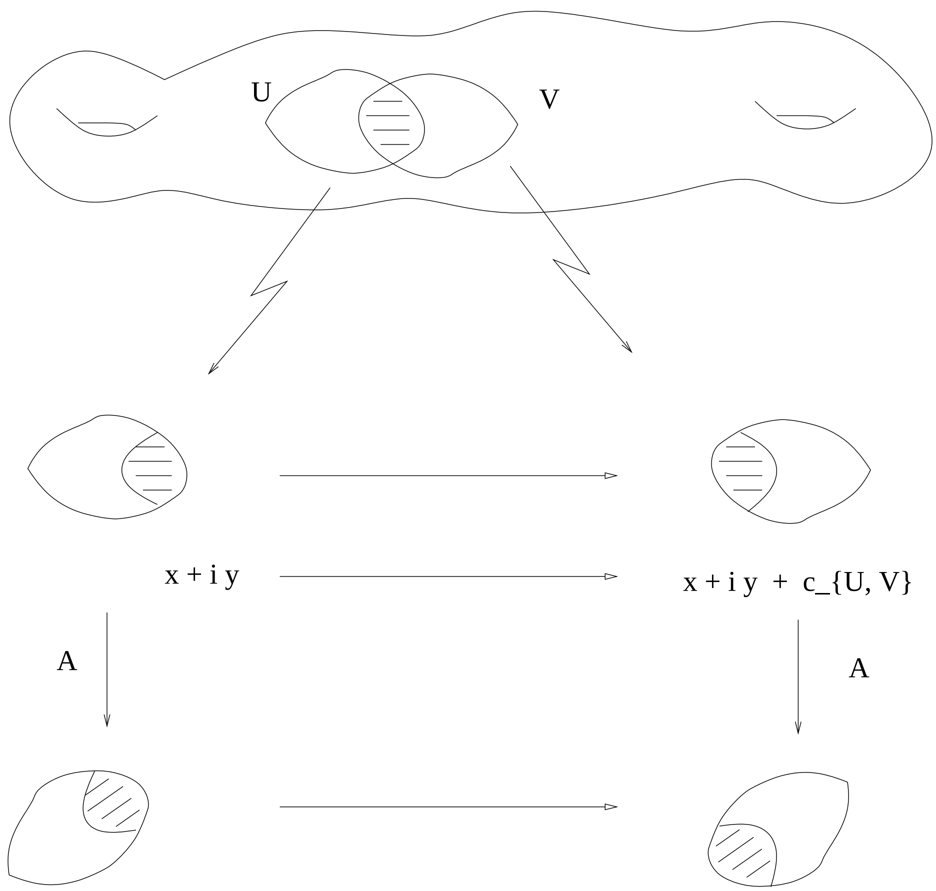
Change base points, coordinate change is by translation:

$$c := \int_{p_0}^{p} \omega - \int_{p_1}^{p} \omega = \int_{p_0}^{p_1} \omega \,.$$

Translation Surface <— —> (X, ω)

II. SL_2R- Action

Post-compose with $A \in SL_2R$.



New translation surface, therefore have action on ΩM_g

Affine Diffeomorphisms

$$(x, y) \longrightarrow (x, x + y \bmod 1)$$

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

An *affine diffeomorphism* is some $f : X \to X$

everywhere (off of singularities) derivative of f is $A \in SL_2R$

Group of all such is stabilizer, the Veech group: $SL(X, \omega)$.

Veech Dichotomy

A *lattice* in $SL(2,R)$ is a subgroup giving a finite area quotient of the hyperbolic plane.

Veech 1989: Let (X,ω) be a translation surface. Suppose $SL(X,\omega)$ is a lattice in $SL(2,R)$. Then for each direction θ, the flow F_θ is either periodic or uniquely ergodic.

McMullen 2006+: In genus two, the converse holds.

Infinitely Generated Veech Groups

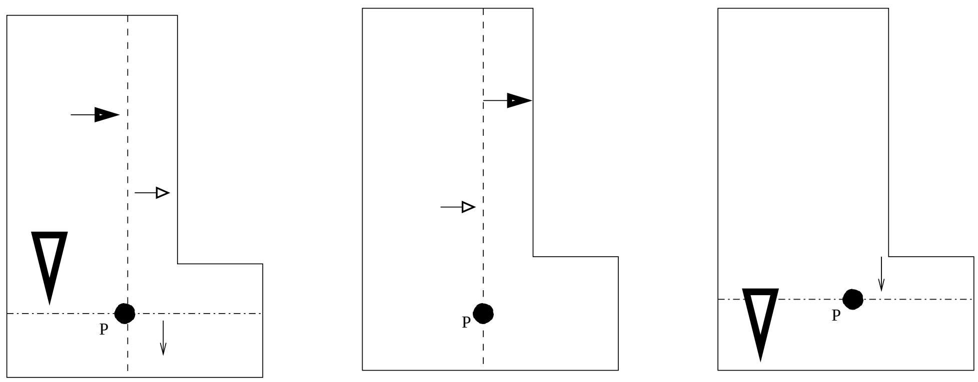
Veech 1995: Do infinitely generated $SL(X, \omega)$ exist?

Hubert-S. 2004: Infinitely generated $SL(X, \omega)$ exist for genus four or greater.

McMullen 2003: In genus two, infinitely generated $SL(X, \omega)$ exist.

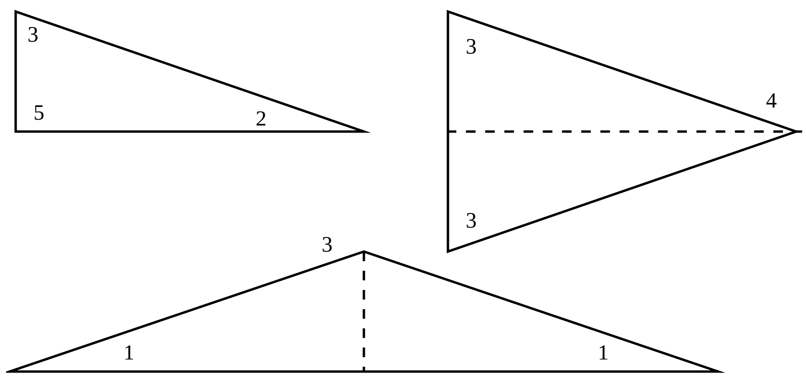
Smillie-Weiss 2006+: For certain Hubert-S. examples, converse of Veech Dichotomy does not hold.

Ramified Over 1 Point



Ramification over a single point.

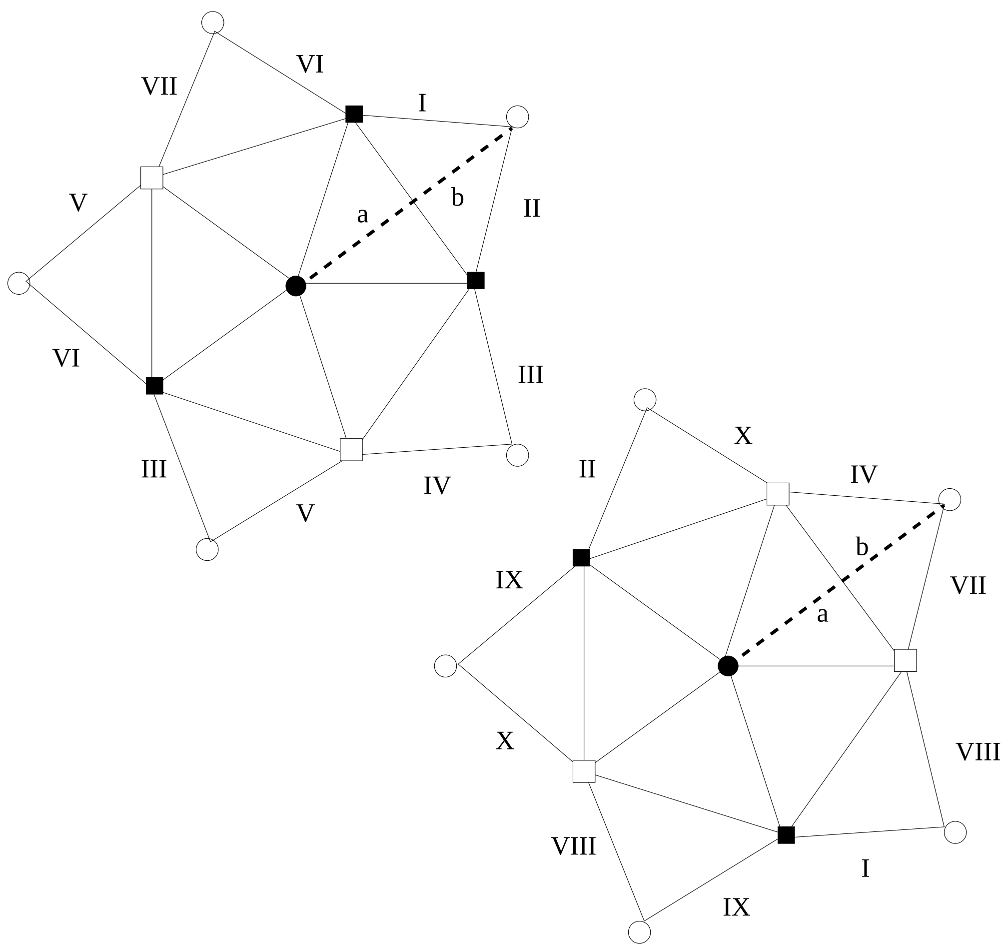
Ramified Cover: Related Triangles 1



Right angled triangle gives same as $(\pi/5, \pi/5, 3\pi/5)$, whereas $(3\pi/10, 3\pi/10, 2\pi/5)$ gives cover.

Ramified Cover: Related Triangles 2

A ramified cover.



Surface from $(3, 3, 4)$ double covers $(1, 1, 3)$ surface

III. Geometry: Special Points

Two direct analogs of points of rational coordinates on square torus.

- If P has finite orbit under $\mathrm{Aff}^{+}(X, \omega)$, we call P *periodic*.

  Gutkin-Hubert-S. 2003; Möller 2006+: If $SL(X, \omega)$ is a "nonarithmetic" lattice group, then there are only finitely many periodic points.

- If P is periodic under two parabolic elements corresponding to tranverse fixed directions, we call P *rational*.

Connection Points

- A geodesic line emanating from a singularity is a *separatrix*.
- A separatrix joining singularities (with no other singularities included) is called a *saddle connection*.

If P is such that any separatrix passing through P can be extended to a saddle connection, we call P a *connection point*.

A result

Hubert-S. 2004, '06 There exists (Y, α) with non-periodic connection points. Suppose $X \to Y$ is a ramified cover of Riemann surfaces, branched above one such point; let $\omega = f^*\alpha$. Then

- $SL(X, \omega)$ is infinitely generated
- $SL(X, \omega)$ uniformizes a surface with infinitely many cusps.
- The projection to M_g of the $SL_2(R)$-orbit of (X, ω) is dense in an algebraic surface.

IV. Infinitely Generated: Limit Sets

- Veech 1989: Any $SL(X, \omega)$ is a Fuchsian group.
- The limit points of a Fuchsian group Γ acting on the hyperbolic upper half-plane must actually lie on the boundary.
- If this set is the full boundary, Γ is either a lattice or else is infinitely generated.

Ramified Covers

From results of Gutkin-Judge 1996/2000 and Vorobets 1996,

$SL(X, f^*\alpha)$ is commensurable to the stabilizer in $SL(X, \alpha)$ of the branch locus of f.

Thus, if branching is over non-periodic point, $SL(X, f^*\alpha)$ cannot be a lattice.

Role of Connection Points

- Veech 1989: Direction of saddle connection on (Y, α) of lattice group is fixed by a parabolic element.
- Parabolic element fixes the saddle connections of its direction.
- Directions of separatrices passing through any point is dense in circle.
- If P is a connection point on a surface with lattice Veech group, then it lies on dense set of saddle connections. It hence has dense parabolic directions associated to its stabilizer.

Connection Points Exist

Key step — show that there are surfaces with non-periodic connection points.

- In fact, the $(1, 1, 3)$-surface is one such.
- With work of K. Calta 2004 and McMullen 2003, there are known to be many such.

V. Infinitely Many Cusps: Set-up

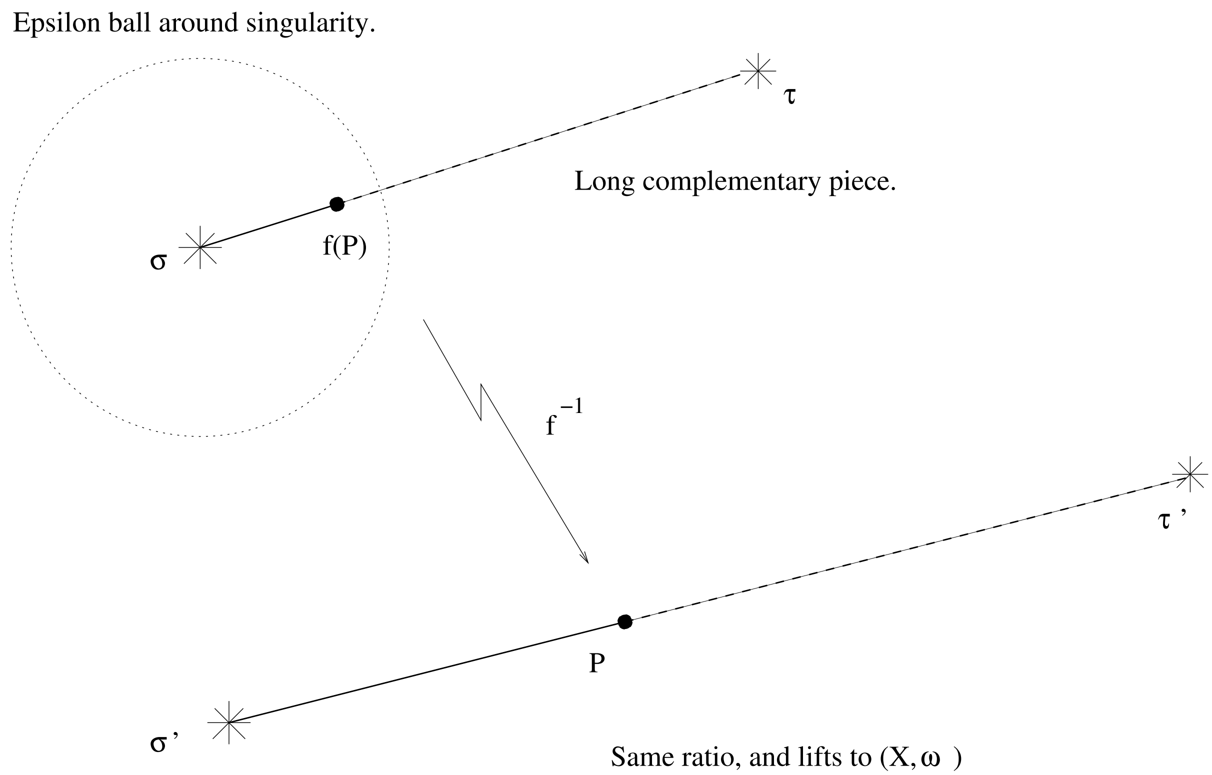
- A cusp of a the hyperbolic surface $\Gamma\backslash H$ corresponds to a (maximal) parabolic conjugacy subgroup in the Fuchsian group Γ.

- Ratio of lengths of saddle connections lying in same direction is invariant under $SL(X, \omega)$.

- Gutkin-Hubert-S. 2003. Non-periodic points of a lattice surface have dense orbit.

- There is a lower bound on the length of saddle connections on (Y, α).

Sketch



Arbitrarily large ratio — therefore infinitely many cusps.

VI. Projects to be Dense in Algebraic Surface: Comparison

McMullen 2003: In genus two, $SL(X, \omega)$ infinitely generated implies that $SL_2(R)$-orbit projects to be dense in Hilbert modular surface.

Compare also with: Eskin, Marklof and Morris, *Unipotent flows and branched covers of Veech surfaces*, to appear.

Sketch

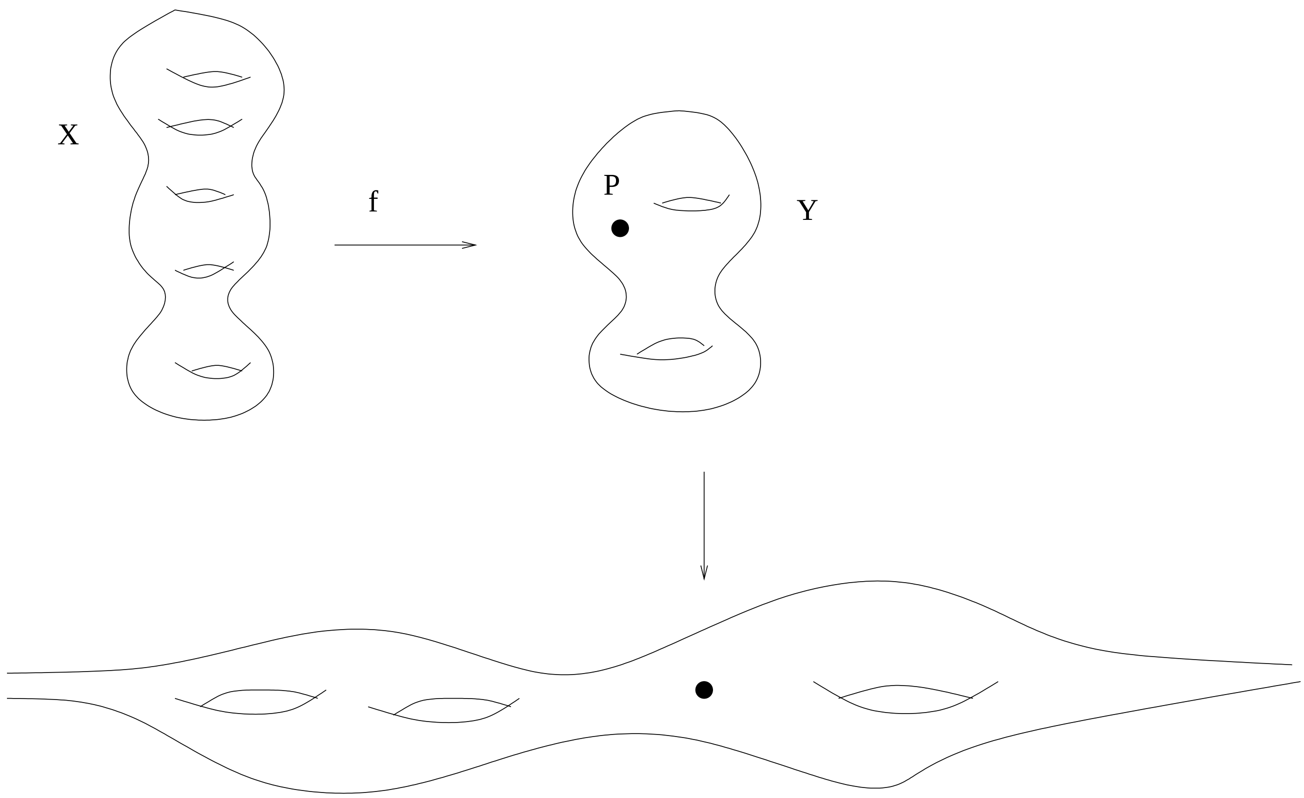
The idea of proof is reasonably easy —

The projection of the $SL_2(R)$-orbit of (Y, α) is an algebraic curve, in Riemann moduli space. Up to normalizing singularities, each point of this curve has a corresponding Riemann surface fibering over it. This fibered complex surface should parametrize classes of covers ramified over one point of the corresponding Riemann surfaces (with fixed ramification structure).

Thus the biholomorphic equivalence class of our X will lie in (some component) of this surface. But, the density of the orbit of p leads to the density of the projection of the $SL_2(R)$-orbit of (X, ω) in this surface.

One way to turn this into a proof is to use the following.

Relating to Universal Curve

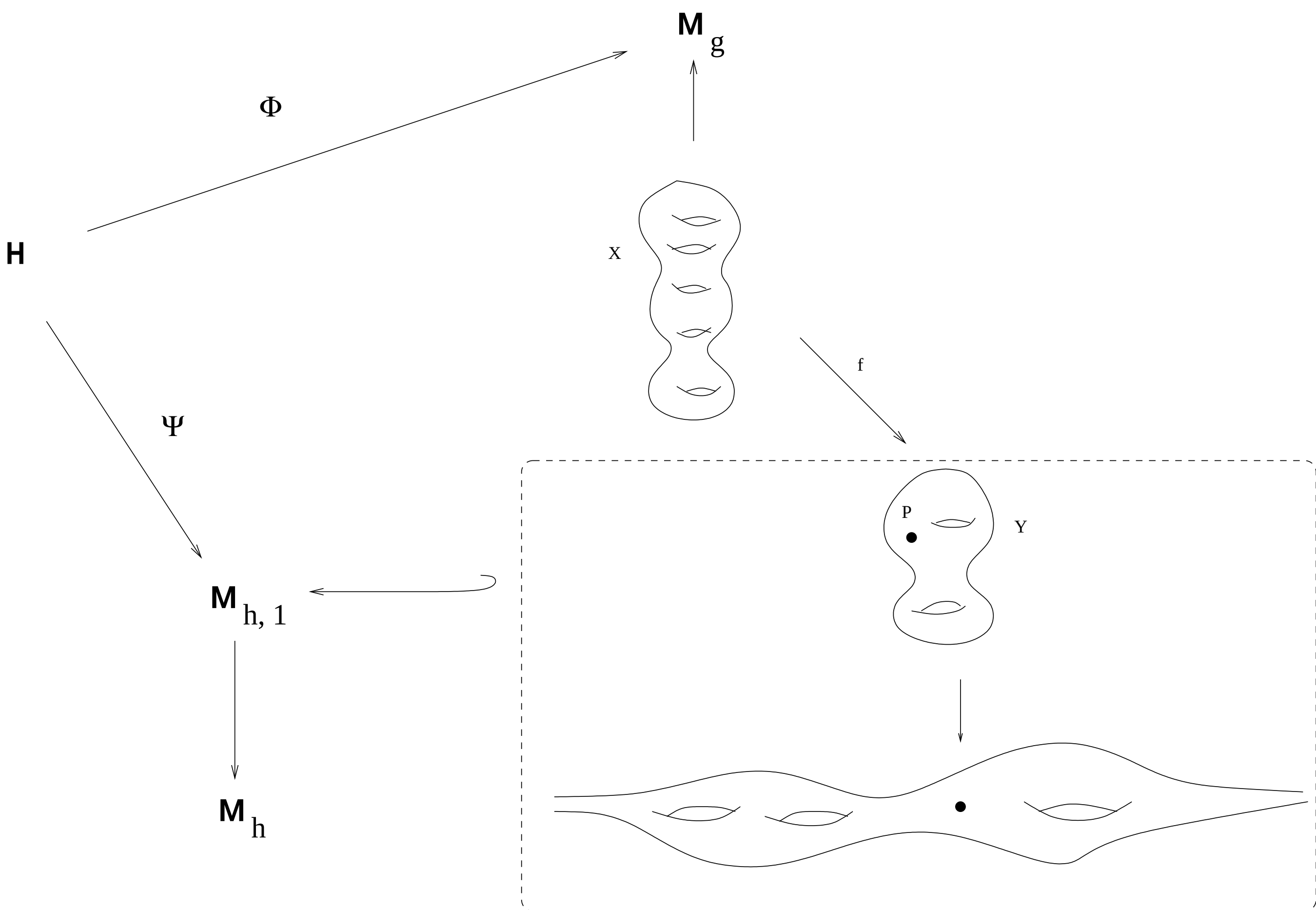


Ramified covering, projection to Teichmüller curve in M_h.

Hurwitz Spaces

Wewers' Thesis 1998 Fix a genus h and a natural number r. Suppose that G is a monodromy group realized by some cover $\rho : X \to Y$ with Y of genus h, ρ having r branch points, and X say of genus g. Then there is a quasi-projective space $\mathcal{H}$, parametrizing all covers of genus h Riemann surfaces with monodromy group G, with finite degree, (forgetful) algebraic maps $\Phi : \mathcal{H} \to \mathcal{M}_g$ and $\Psi : \mathcal{H} \to \mathcal{M}_{h,[r]}$.

Picturing Hurwitz



Surface in $M_{h,1}$ leads to surface in M_g.