Gonality of modular curves in characteristic p

Bjorn Poonen

University of California at Berkeley

April 30, 2006

Definition of gonality

- Let X be a smooth, projective, geometrically integral curve of genus g over a field k.
- Define the gonality of X as

$$\gamma_k(X) \;:=\; \min\{\, \deg \pi \mid \pi \colon X \twoheadrightarrow \mathbb{P}^1_k \,\}.$$

- If $L \supseteq k$, let $\gamma_L(X)$ be the gonality of $X_L := X \times_k L$.

Properties of gonality

Proposition

1. *If* $L \supseteq k$*, then* $\gamma_L(X) \leq \gamma_k(X)$*. If* $k = \overline{k}$*, equality holds.*
2. *If* $g > 1$*, then* $\gamma_k(X) \leq 2g - 2$*.*
3. *If* $g > 1$ *and* $X(k) \neq \emptyset$*, then* $\gamma_k(X) \leq g$ *(Riemann-Roch).*
4. *If* $k = \overline{k}$*, then* $\gamma_k(X) \leq \lfloor \frac{g+3}{2} \rfloor$ *(Brill-Noether theory).*
5. *If* $\pi \colon X \twoheadrightarrow Y$*, then* $\gamma_k(Y) \leq \gamma_k(X) \leq (\deg \pi)\gamma_k(Y)$*.*

All these bounds are best possible: for 2. and 3. this follows from the *Franchetta conjecture* and its generalizations, which in turn follow from the calculation of $\operatorname{Pic}(\mathcal{M}_{g,n})$ for $n \leq 2$. (The Franchetta conjecture, proved by Harer in 1983, states that the Picard group of the general curve over the function field of M_g is generated by the canonical class.)

Bounding gonality change under field extension

The following is implicit in a 1996 preprint of K. V. Nguyen and M.-H. Saito.

Theorem

Let k be a perfect field. Let X/k be a curve with a k-point. Then $\gamma_L(X) \geq \sqrt{\gamma_k(X)}$ for any $L \supseteq k$.

Remark: The hypothesis that X has a k-point is essential: genus-1 curves over $\mathbb{Q}$ can have arbitrarily high gonality, but all have $\overline{\mathbb{Q}}$-gonality 2.

Classical modular curves

Suppose char $k \nmid N$.

- The Galois module $\mu_N \times \mathbb{Z}/N\mathbb{Z}$ is equipped with a nondegenerate alternating pairing taking values in μ_N.
- Similarly, if E is an elliptic curve, then $E[N]$ is equipped with the Weil pairing.
- Let $X(N)$ be the smooth projective model of the moduli space of pairs (E, ϕ) where $\phi \colon \mu_N \times \mathbb{Z}/N\mathbb{Z} \to E[N]$ is a pairing-preserving isomorphism.
- Now assume $k = \overline{k}$. The action of $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ on $\mu_N \times \mathbb{Z}/N\mathbb{Z}$ induces an action of $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm 1\}$ on $X(N)$.
- If $G \le \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm 1\}$, let $X_G := X(N)/G$.

Igusa and hybrid curves

Suppose char $k = p$ and $p \nmid N$.

- The Igusa curve $\mathrm{Ig}(p^e)$ is the smooth projective model of the moduli space of pairs (E, R) where E is an ordinary elliptic curve and R is a generator of the kernel of the p^e-Verschiebung $V\colon E^{(q)} \to E$.
- The hybrid curve $X(p^e; N)$ is the smooth projective model of the moduli space of (E, ϕ, R) where ϕ is as in the definition of $X(N)$, and R is as in the definition of $\mathrm{Ig}(p^e)$.
- $X(p^e; N) \to X(1)$ is generically Galois with group $((\mathbb{Z}/p^e\mathbb{Z})^\times \times \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}))/\{\pm 1\}$.
- They fit in an inverse system with profinite Galois group

$$S := \frac{\mathbb{Z}_p^\times \times \prod_{\ell \neq p} \mathrm{SL}_2(\mathbb{Z}_\ell)}{\{\pm 1\}}.$$

- For any open subgroup $G \leq S$, we can define a general hybrid curve X_G as a quotient of some $X(p^e; N)$.

Lower bounds on gonality of modular curves

Our goal is to prove lower bounds on gonality of modular curves. It suffices to consider $k = \mathbb{C}$ and $k = \overline{\mathbb{F}}_p$.

Theorem (Abramovich 1996)

$\gamma_{\mathbb{C}}(X_G) \geq \frac{7}{800}(S : G)$

The proof is analytic: it uses lower bounds on the leading nontrivial eigenvalue of the noneuclidean Laplacian, and the notion of *conformal area* developed by P. Li and S.-T. Yau.

In characteristic p, we have the following, generalizing work of Ogg, Harris-Silverman, Nguyen-Saito, Baker, Schweizer:

Theorem (P.)

Fix p. Then as G ranges through open subgroups of S, we have $\gamma_{\overline{\mathbb{F}}_p}(X_G) \to \infty$.

We next sketch the proof of this theorem.

Lower bounds for classical modular curves: Ogg's strategy

- For any $X/\mathbb{F}_q$, we have $\gamma_{\mathbb{F}_q}(X) \geq \frac{\#X(\mathbb{F}_q)}{\#\mathbb{P}^1(\mathbb{F}_q)}$.
- This gives nontrivial lower bounds on $\gamma_{\mathbb{F}_{p^2}}(X_0(N))$ since the number of supersingular points on $X_0(N)$ goes to ∞ with N, and they are all defined over $\mathbb{F}_{p^2}$.
- Then $\gamma_{\overline{\mathbb{F}}_p}(X_0(N)) \geq \sqrt{\gamma_{\mathbb{F}_{p^2}}(X_0(N))} \to \infty$.

The same argument works for $X(N)$, provided that one uses a twisted form of $X(N)$:

- Let M be the $\operatorname{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^2})$-module $(\mathbb{Z}/N\mathbb{Z})^2$ with $\operatorname{Frob}_{p^2}$ acting as multiplication by $-p$.
- Let $X(N)'$ be the curve over $\mathbb{F}_{p^2}$ parameterizing (E, ϕ) where $\phi\colon M \to E[N]$ is a pairing-preserving isomorphism.
- Then all supersingular points on $X(N)'$ are over $\mathbb{F}_{p^2}$.

Lower bounds for Igusa curves

For $\mathrm{Ig}(q)$ the above method with supersingular points does not work, because there are not enough (they ramify totally in the Igusa tower).

But there are several alternatives:

- Show that $\mathrm{Ig}(q)$ has a lot of ordinary points over $\mathbb{F}_q$! Namely, there are at least $q^{3/2-o(1)}$ (Pacheco).
- Show that $\mathrm{Ig}(q)$ has a lot of cusps over $\mathbb{F}_p$: they split completely in the Igusa tower.
- Show that the existence of the map $\mathrm{Ig}(q) \to \mathrm{Ig}(1)$, whose degree is low relative to the genus of $\mathrm{Ig}(q)$, rules out the existence of other low-degree maps to $\mathbb{P}^1$ (Schweizer).

Lower bounds for hybrid curves

To prove the result for all modular curves:

1. Use Goursat's lemma to understand the subgroups of

$$\mathbb{Z}_p^\times \times \prod_\ell \mathrm{SL}_2(\mathbb{Z}_\ell).$$

2. Deduce that a modular curve with large index $(S : G)$ dominates either a classical modular curve of large index or an Igusa-type curve of large index.

Image of Galois

By the work of Mazur, Kamienny, and Merel, we have

Theorem (Strong uniform boundedness of torsion of elliptic curves over number fields)

$\forall d \geq 1, \exists N_d, \forall [K:\mathbb{Q}] \leq d, \forall E/K, \#E(K)_{\text{tors}} \leq N_d.$

For $[K:\mathbb{Q}] < \infty$ and E/K, we get $\rho \colon \operatorname{Gal}(\overline{K}/K) \to \operatorname{GL}_2(\hat{\mathbb{Z}})$.

Conjecture (image of Galois over number fields)

$\forall d \geq 1, \exists N_d, \forall [K:\mathbb{Q}] \leq d, \forall E/K$ *without CM*, $(\operatorname{GL}_2(\hat{\mathbb{Z}}) : \rho(\operatorname{Gal}(\overline{K}/K))) \leq N_d$.

Our gonality bounds prove the function field analogue:

Theorem (image of Galois over function fields)

$\forall p \geq 0, \forall d \geq 1, \exists N_{p,d}, \forall \operatorname{char} k = p, \forall [K:k(t)] \leq d,$ $\forall E/K$ *with* $j(E) \notin \overline{k}$, $\rho(\operatorname{Gal}(K^s/K))$ *contains a subgroup of index at most* N_d *in* $\mathbb{Z}_p^\times \times \prod_{\ell \neq p} \operatorname{SL}_2(\mathbb{Z}_\ell)$.

For further details, see my article at

`http://math.berkeley.edu/~poonen/papers/gonality.pdf`