

Traces of singular moduli on Hilbert modular surfaces

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Class Number Generating Function.

Theorem. (Gauss)

If $H(d)$ is the Hurwitz-Kronecker class number of discriminant $-d$ binary quadratic forms and

$$\sum_{x,y,z \in \mathbb{Z}} q^{x^2+y^2+z^2} = \sum_{n=0}^{\infty} r(n)q^n,$$

then

$$r(n) = \begin{cases} 12H(4n) & \text{if } n \equiv 1, 2 \pmod{4}, \\ 24H(n) & \text{if } n \equiv 3 \pmod{8}, \\ r(n/4) & \text{if } n \equiv 0 \pmod{4}, \\ 0 & \text{if } n \equiv 7 \pmod{8}. \end{cases}$$

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Remark. This generating function is a weight $3/2$ modular form.

Singular moduli

The j -function.

Throughout let $q := e^{2\pi iz}$, and as usual let

$$j(z) = q^{-1} + 744 + 196884q + 21493760q^2 + \dots .$$

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Definition. Values of $j(z)$ at imaginary quadratic arguments in \mathfrak{H} are known as **singular moduli**.

Classical Examples.

$$j(i) = 1728, \quad j\left(\frac{1 + \sqrt{-3}}{2}\right) = 0,$$

$$j\left(\frac{1 + \sqrt{-15}}{2}\right) = \frac{-191025 - 85995\sqrt{5}}{2}.$$

Theorem.

Singular moduli are algebraic integers.

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Remark. Singular moduli have many roles.

- Generate class fields of imag. quad. fields.
- Explain the interplay between elliptic curves over finite fields and CM elliptic curves.
- Provide structure for Borcherds' work on infinite product expansions of modular forms.

Traces of singular moduli

Definition. The *discriminant $-d$ trace* is

$$\mathrm{Tr}(d) := \sum_{Q \in \mathcal{Q}_d/\Gamma} \frac{j(\alpha_Q) - 744}{\omega_Q},$$

where

$$\omega_Q := \begin{cases} 2 & \text{if } Q \sim_{\Gamma} [a, 0, a], \\ 3 & \text{if } Q \sim_{\Gamma} [a, a, a], \\ 1 & \text{otherwise.} \end{cases}$$

Remark. The integer $\mathrm{Tr}(d)$ is the trace of algebraic conjugates

$$\frac{j(\alpha_Q) - 744}{\omega_Q}.$$

Singular moduli generating function

Theorem. (Zagier)

The generating series

$$-q^{-1} + 2 + \sum_{0 < d \equiv 0,3 \pmod{4}} \text{Tr}(d)q^d$$

is a weight $3/2$ modular form.

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Question. What does this lead to?

Answer. Zagier obtained Borcherds' infinite product theorem for modular forms with Heegner divisor.

Hilbert modular surfaces.

Question. Are “singular moduli” generating functions modular in other settings?

Answer. Yes.

Notation.

- Let $p \equiv 1 \pmod{4}$ be prime.
- Let $\mathcal{O}_p := \mathbb{Z} \left[\frac{1+\sqrt{p}}{2} \right]$, the integers of $\mathbb{Q}(\sqrt{p})$.
- Let \mathfrak{h} denote the complex upper half plane.

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Fact. The group $SL_2(\mathcal{O}_p)$ acts on $\mathfrak{h} \times \mathfrak{h}$

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \circ (z_1, z_2) := \left(\frac{\alpha z_1 + \beta}{\gamma z_1 + \delta}, \frac{\alpha' z_2 + \beta'}{\gamma' z_2 + \delta'} \right).$$

Here ν' denotes the conjugate of ν in $\mathbb{Q}(\sqrt{p})$.

We consider the modular surface

$$X_p := (\mathfrak{h} \times \mathfrak{h}) / SL_2(\mathcal{O}_p).$$

“A Modular Intersection Theorem.”

Hirzebruch-Zagier.

Define algebraic curves $Z_1^{(p)}, Z_2^{(p)}, \dots \subset X_p$ by:

1. For a positive integer N , consider the points $(z_1, z_2) \in \mathfrak{h} \times \mathfrak{h}$ satisfying an equation of the form

$$Az_1z_2\sqrt{p} + \lambda z_1 - \lambda' z_2 + B\sqrt{p} = 0,$$

where $A, B \in \mathbb{Z}$, $\lambda \in \mathcal{O}_p$, and $\lambda\lambda' + ABp = N$.

2. The *Hirzebruch-Zagier divisor* $Z_N^{(p)}$ is this union in X_p .

Facts.

- Each equation defines a curve in $\mathfrak{h} \times \mathfrak{h}$ isomorphic to \mathfrak{h} , and their union is invariant under $SL_2(\mathcal{O}_p)$.
- If $\left(\frac{N}{p}\right) = -1$, then $Z_N^{(p)}$ is empty.

Notation.

- Let $\widetilde{Z}_N^{(p)}$ denote the closure of $Z_N^{(p)}$ in Y_p , the “resolved” X_p .
- Let $(\widetilde{Z}_m^{(p)}, \widetilde{Z}_n^{(p)})$ be the intersection number of $\widetilde{Z}_m^{(p)}$ and $\widetilde{Z}_n^{(p)}$.

Theorem. (H-Z, Invent. Math. '76)

$$\Phi_m^{(p)}(z) := a_m^{(p)}(0) + \sum_{n=1}^{\infty} (\widetilde{Z}_m^{(p)}, \widetilde{Z}_n^{(p)}) q^n$$

is a holomorphic weight 2 modular form on $\Gamma_0(p)$ with Nebentypus $\left(\frac{\cdot}{p}\right)$.

Remarks.

1. The proof describes

$$\widetilde{Z}_m^{(p)} \cap \widetilde{Z}_n^{(p)}$$

in terms of $\Gamma_0(m)$ Heegner points with discriminants of the form $-(4mn - x^2)/p$.

2. So, the HZ theorem is a “singular moduli” theorem for the modular function $f(z) = 1$.

Question. Is there a generalization for “generic” singular moduli?

Answer. Yes.

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Notation and Hypotheses.

- Suppose that $\ell = 1$ or is an odd prime.
- Let $f(z) = \sum a(n)q^n$ be a modular function on $\Gamma_0^*(\ell)$ with $a(0) = 0$.

Definition. Define the “trace”

$$(Z_\ell^{(p)}, Z_n^{(p)})_f^{\text{tr}} := \sum_{\tau \in Z_\ell^{(p)} \cap Z_n^{(p)}} \frac{f(\tau)}{\#\Gamma_0^*(\ell)_\tau},$$

where $\Gamma_0^*(\ell)_\tau$ denotes the stabilizer of τ in $\Gamma_0^*(\ell)$.

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Define the “generating function”

$$\Phi_{\ell,f}^{(p)}(z) := A_{\ell,f}^{(p)}(z) + B_{\ell,f}^{(p)}(z) + \sum_{n=1}^{\infty} (Z_\ell^{(p)}, Z_n^{(p)})_f^{\text{tr}} q^n,$$

where

$$A_{\ell,f}^{(p)}(z) := -\epsilon(\ell) \sum_{m,n \geq 1} ma(-mn) \cdot \left(\sum_{\substack{x \in \mathbb{Z} \\ x^2 \equiv m^2 p \pmod{2\ell}}} q^{\frac{x^2 - m^2 p}{4\ell}} + \sum_{\substack{x \in \mathbb{Z} \\ x \equiv m \pmod{2}}} q^{\frac{x^2 \ell - m^2 p \ell}{4}} \right),$$

$$B_{\ell,f}^{(p)}(z) := 2\epsilon(\ell) \sum_{n \geq 1} (\sigma_1(n) + \ell \sigma_1(n/\ell)) a(-n) \sum_{x \in \mathbb{Z}} q^{\ell x^2},$$

where $\epsilon(\ell) = 1/2$ for $\ell = 1$, and is 1 otherwise.

Thm 1. (*Bringmann, O, Rouse, IMRN '05*)

Assume the notation above. If $\left(\frac{\ell}{p}\right) \neq -1$, then

$$\Phi_{\ell,f}^{(p)}(z) \in \mathcal{M}_2 \left(\Gamma_0(pl^2), \left(\frac{\cdot}{p}\right) \right).$$

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Remarks. The proof requires:

- Work of Bruinier and Funke on “theta lifts” ,
- Facts about Heegner points on $X_0(m)$,
- Modifications of H-Z arguments,
- Facts about modular forms.

Special case of $J_1(z) = j(z) - 744$.

Question. If $f(z) = J_1(z) := j(z) - 744$, we have that

$$\Phi_{1, J_1}^{(p)}(z) \in \mathcal{M}_2(\Gamma_0(p), \left(\frac{\cdot}{p}\right)).$$

How do these forms vary with p ?

- In terms of Dedekind's eta-function

$$\eta(z) := q^{1/24} \prod_{n=1}^{\infty} (1 - q^n),$$

and the classical Weber functions

$$f_1(z) = \frac{\eta(z/2)}{\eta(z)} \quad \text{and} \quad f_2(z) = \sqrt{2} \cdot \frac{\eta(2z)}{\eta(z)}?$$

- In terms of Hilbert class polynomials?

Thm 2. (*Bringmann, O, Rouse, IMRN '05*)

We have that

$$\Phi_{1,J_1}^{(p)}(z) = \frac{\eta(2z)\eta(2pz)E_4(pz)f_2(2z)^2f_2(2pz)^2}{4\eta(pz)^6} \cdot \left(f_1(4z)^4f_2(z)^2 - f_1(4pz)^4f_2(pz)^2 \right).$$

Remarks. To prove Thm 2, one needs:

- Properties of the Borcherds forms,
- Zagier duality between weight $1/2$ and $3/2$ “nearly” holomorphic modular forms in Kohnen’s plus space,
- Modular form operator facts.

Hilbert class polynomials.

If $-D$ is a fundamental discriminant, then let

$$H_D(x) = \prod_{\tau \in \mathcal{C}_D} (x - j(\tau)) \in \mathbb{Z}[x].$$

Fact. $H_D(x)$ is irreducible and generates the Hilbert class field of $\mathbb{Q}(\sqrt{-D})$.

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Definition. Define $N_p(z)$ by

$$N_p(z) := \prod_{M \in \Gamma_0(p) \backslash \text{SL}_2(\mathbb{Z})} \Phi_{1, J_1}^{(p)} | M.$$

Examples.

$$N_p(z) = \begin{cases} \Delta \cdot H_{75}(j(z)) & \text{if } p = 5, \\ E_4 \Delta^2 \cdot H_3(j(z)) H_{507}(j(z)) & \text{if } p = 13, \\ \Delta^3 \cdot H_4(j(z)) H_{867}(j(z)) & \text{if } p = 17, \\ \Delta^5 \cdot H_7(j(z))^2 H_{2523}(j(z)) & \text{if } p = 29. \end{cases}$$

Notation.

- Define integers $a(p)$, $b(p)$, and $c(p)$ by

$$a(p) = \frac{1}{2} \left(\left(\frac{3}{p} \right) + 1 \right), b(p) = \frac{1}{2} \left(\left(\frac{2}{p} \right) + 1 \right),$$

$$c(p) = \frac{1}{6} \left(p - \left(\frac{3}{p} \right) \right).$$

- Let \mathcal{D}_p be the set of negative discriminants $-D \neq -3, -4$ of the form $\frac{x^2 - 4p}{16f^2}$.

Thm 3. (*Bringmann, O, Rouse, IMRN '05*)
In the notation above, we have

$$N_p(z) = (E_4 H_3(j(z)))^{a(p)} H_4(j(z))^{b(p)} \Delta^{c(p)} \\ \cdot H_{3,p^2}(j(z)) \cdot \prod_{-D \in \mathcal{D}_p} H_D(j(z))^2.$$

Remark. To prove Thm 3, one simply:

- First observes that $N_p(z)$ is on $SL_2(\mathbb{Z})$ since it is merely a “multiplicative norm”.
- One computes the weight and certain “special points” in the divisor of $N_p(z)$.
- Apply the irreducibility of $H_D(x)$ together with the Eichler-Selberg trace formula to complete the factorization.

Summary

Remarks.

Generating functions for traces of singular moduli on Hilbert modular surfaces

- Are often weight 2 modular forms,
- For $j(z) - 744$ are given in terms of Weber functions,
- For $j(z) - 744$ are products of Hilbert class polynomials.