Hilbert modular generating functions with coefficients in intersection homology

Jayce Getz, UW-Madison
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(joint work with Mark Goresky).

Notation for Hirzebruch-Zagier:

Let $p \equiv 1 \pmod 4$ be a prime ($p > 0$).

- $Y_p := \mathsf{SL}_2(\mathcal{O}_{\mathbb{Q}(\sqrt{p})})\backslash\mathfrak{H}^2$

- $X_p :=$ Satake-Bailey-Borel compactification of Y_p

- $X_p^T :=$ a particular (smooth) toroidal compactification of Y_p.

$$Z_m^\circ : \; = \; \pi\left(\left\{(z_1, z_2) \in \mathfrak{H}^2 : \begin{array}{r} z_2 = \frac{\gamma' z_1 - b\sqrt{p}}{a\sqrt{p} z_1 + \gamma} \\ a, b \in \mathbb{Z}, \gamma \in \mathcal{O}_{\mathbb{Q}(\sqrt{p})} \\ \gamma\gamma' + abp = m \end{array}\right\}\right)$$

where

$$\pi : \mathfrak{H}^2 \to Y_p$$

is the canonical projection.

Let

$$\Phi := \sum_{n=0}^{\infty} [Z_n] q^n.$$

Theorem (Hirzebruch-Zagier Invent. ‘76)

For each $\xi \in H_2(X_p^T)$ the formal Fourier expansion

$$\langle \xi, \Phi \rangle_H := \sum_{n \geq 0} \langle \xi, [Z_n] \rangle_H \ q^n$$

is a weight 2 elliptic modular form with nebentypus.

Theorem (Zagier, LNM '76):

$$\begin{aligned}\langle [Z_m], \Phi \rangle_H &= c(m) E_{2,p}(z) \\ &+ r \sum_{n=1}^{\infty} \left(\sum_f {}' \frac{\left(\int_{Z_1} \eta_{\widehat{f}} \right)^2}{(\widehat{f}, \widehat{f})} a_f(m) a_f(n) \right) q^n.\end{aligned}$$

. . . .

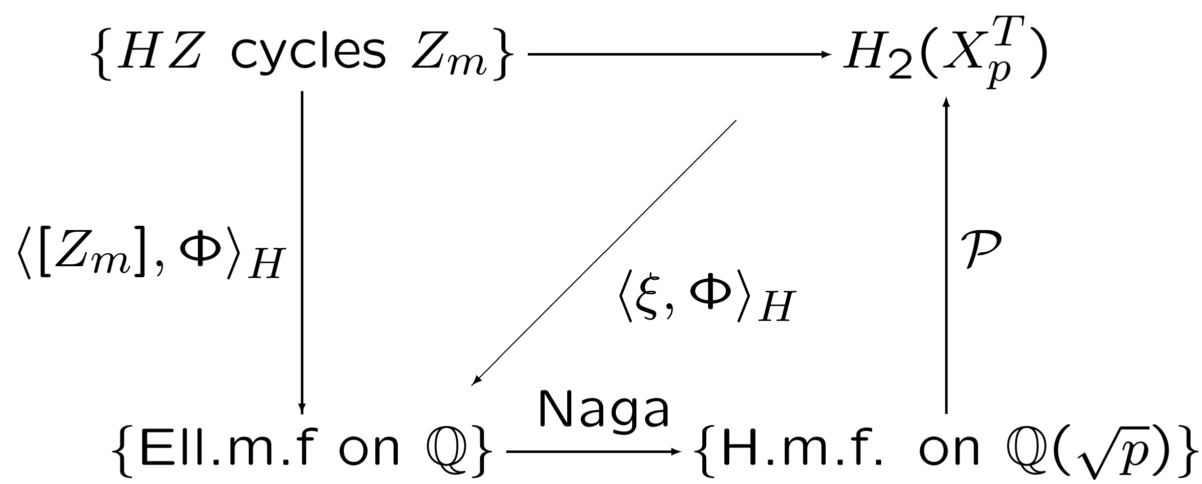
- $'$ indicates we sum over a basis of weight two newforms for $\Gamma_0(p)$ with character $\left(\frac{p}{\cdot}\right)$.

- $E_{2,p} :=$ an Eisenstein series for this space.
  $r :=$ constant
  $c(m) :=$ constant depending on m.

- $(,) :=$ Petersson inner product.

- $\widehat{f}$ is the Naganuma lift of f
  $\eta_{\widehat{f}} :=$ a differential form attached to $\widehat{f}$.

Relationship with the Naganuma lifting



Notation:

- $L/E :=$ quadratic extension of totally real number fields with associated Hecke character η.
- $\Sigma(L) :=$ set of embeddings $\sigma : L \hookrightarrow \mathbb{R}$.
- $\mathcal{O}_L :=$ ring of integers of L.
- $\mathfrak{c} :=$ ideal of $\mathcal{O}_L$.
- $\mathfrak{c}_E :=$ ideal of $\mathcal{O}_E$.
- $G := G_L := \mathrm{Res}_{L/\mathbb{Q}}(\mathrm{GL}_2)$.

Hilbert modular varieties

$$\begin{aligned} Y_0(\mathfrak{c}) : &= G(\mathbb{Q})\backslash G(\mathbb{A})/K_\infty K_0(\mathfrak{c}). \\ X_0(\mathfrak{c}) : &= \text{Satake-Bailey-Borel of } Y_0(\mathfrak{c}). \end{aligned}$$

- K_∞ is the stabilizer of

$$\begin{array}{ccc} \mathbb{C}^\times & \to & G(\mathbb{R}) \\ x+iy & \mapsto & \left(\begin{pmatrix} x & y \\ -y & x \end{pmatrix}, \ldots, \begin{pmatrix} x & y \\ -y & x \end{pmatrix}\right). \end{array}$$

- $K_0(\mathfrak{c})$ is the compact open

$$\{\gamma \in \mathrm{GL}_2(\widehat{\mathcal{O}_L}) : \gamma \equiv \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \pmod{\mathfrak{c}}\}.$$

Analytic realization:

$$Y_0(\mathfrak{c}) = \bigcup \Gamma_j \backslash \mathfrak{H}^{\Sigma(L)}.$$

Abstracting the situation of HZ:

$$
\begin{array}{ccc}
\{\text{Shimura subv. att. to } E\} & \longrightarrow & IH_\bullet(X_0(\mathfrak{c})) \\
\langle [Z], \Phi\rangle_{IH} \Big\downarrow & \swarrow \langle \xi, \Phi\rangle_{IH} & \Big\uparrow \mathcal{P} \\
\{\text{H.m. forms on } E\} & \xrightarrow{\text{BC}} & \{\text{H.m. forms on } L\}
\end{array}
$$

Compare with:

$$
\begin{array}{ccc}
\{HZ \text{ cycles } Z_m\} & \xrightarrow{\hspace{3em}} & H_2(X_p^T) \\
\Big\downarrow \langle [Z_m], \Phi \rangle_H & \swarrow \langle \xi, \Phi \rangle_H & \Big\uparrow \mathcal{P} \\
\{\text{Ell.m.f on } \mathbb{Q}\} & \xrightarrow{\text{Naga}} & \{\text{H.m.f. on } \mathbb{Q}(\sqrt{p})\}
\end{array}
$$

Notation:

Define $S^{+,0}(\mathfrak{c}_E, \eta^i)$ to be

$$\left\{ g \in S(K_0(\mathfrak{c}_E), \eta^i) : \begin{array}{c} a(\mathfrak{m}, g) = 0 \text{ if } \eta(\mathfrak{m}) = -1 \\ \text{or } \mathfrak{m} + \mathfrak{c}_E = \mathcal{O}_E \end{array} \right\}.$$

$$IH^E(X_0(\mathfrak{c})) := \bigoplus_{\substack{f \in S^{\text{new}}(K_0(\mathfrak{c})): \\ \lambda_f(\mathfrak{P}^\sigma) = \lambda_f(\mathfrak{P}) \text{ a.e.} \\ \forall \sigma \in \text{Gal}(L/E)}} IH_{[L:\mathbb{Q}]}(X_0(\mathfrak{c}))(f).$$

$$IH^{\eta^i}(X_0(\mathfrak{c})) := \bigoplus_{\substack{\text{nebentypus}(g) = \eta^i \\ \widehat{g} \in S(K_0(\mathfrak{c}))}} IH_{[L:\mathbb{Q}]}(X_0(\mathfrak{c}))(\widehat{g}).$$

The Hecke operators $\widehat{T}_{\eta^i}(\mathfrak{m})$:

For each $\mathfrak{m} \subset \mathcal{O}_L$, we define a Hecke operator

$$\widehat{T}_{\eta^i}(\mathsf{N}_{L/E}(\mathfrak{m})) \in \mathbb{T}_{\mathfrak{c}}.$$

For example, $\widehat{T}_{\eta^i}(\mathsf{N}_{L/E}(\mathfrak{P})^r)$ is defined to be

$$\begin{cases} \frac{1}{2}\left(T_{\mathfrak{c}}(\mathfrak{P}^r) + T_{\mathfrak{c}}(\overline{\mathfrak{P}}^r)\right) & \text{if } \mathfrak{p} \text{ splits} \\ T_{\mathfrak{c}}(\mathfrak{P}^r) + \eta^i(\mathfrak{p})\mathsf{N}_{E/\mathbb{Q}}(\mathfrak{p})\widehat{T}_{\eta^i}(\mathfrak{p}^{2r-2}) & \text{if } \mathfrak{p} \text{ is inert} \\ 0 & \text{otherwise} \end{cases}$$

(the rest are defined by multiplicativity).

These operators satisfy the relation

$$\widehat{f}|\widehat{T}_{\eta^i}(\mathfrak{n}) = \lambda_f(\mathfrak{n})\widehat{f}$$

for f of nebentypus η^i if $\mathfrak{n}$ is coprime to $d_{L/E}(\mathfrak{c}\cap\mathcal{O}_E)$ and in the image of the norm map.

Definition of $\gamma_{\eta^i}(\mathfrak{m}')$:

Let

$$Q^{\eta^i} : IH^E(X_0(\mathfrak{c})) \longrightarrow IH^{\eta^i}(X_0(\mathfrak{c}))$$

be the projection.

For each $\gamma \in IH^E(X_0(\mathfrak{c}))$, define

$$\gamma_{\eta^i}(\mathfrak{m}') := \begin{cases} \widehat{T}_{\eta^i}(\mathsf{N}_{L/E}(\mathfrak{m}))_* Q^{\eta^i}\gamma \\ \qquad\qquad\qquad\qquad \text{if } \mathfrak{m}' = \mathsf{N}_{L/E}(\mathfrak{m}), \\ 0 \text{ otherwise.} \end{cases}$$

Theorem 1 (G.-Goresky).

Let L/E be a quadratic extension and $\gamma \in IH^E(X_0(\mathfrak{c}))$.

We then have that

$$\Phi_{\gamma,\eta^i}\left(\left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right)\right) := |y|_{\mathbb{A}_E} \sum_{\substack{\xi \in E^\times \\ 0 \ll \xi}} \gamma_{\eta^i}(\xi y \mathcal{D}_{E/\mathbb{Q}}) q_{0,0}(\xi x, \xi y)$$

is an element of

$$IH^E(X_0(\mathfrak{c})) \otimes S^{+,0}(\mathcal{N}(\mathfrak{c}), \eta^i).$$

Theorem 2 (G.-Goresky).

Suppose that Z is a subanalytic cycle on $X_0(\mathfrak{c})$ admitting a class $[Z] \in IH^E(X_0(\mathfrak{c}_E))$. If

$$\mathfrak{n} + d_{L/E}(\mathfrak{c} \cap \mathcal{O}_E) = \mathfrak{m} + d_{L/E}(\mathfrak{c} \cap \mathcal{O}_E) = \mathcal{O}_E,$$

then the $\mathfrak{n}$th Fourier coefficient of

$$\langle [Z]_{\eta^i}(\mathfrak{m}), \Phi_{[Z],\eta^i} \rangle_{IH}$$

is

$$\frac{1}{4} \sum_{J \subset \Sigma(E)} \sum_{f} \frac{\int_Z \overline{\omega_J(\widehat{f})} \int_Z \omega_J(\widehat{f})}{\int_{Y_0(\mathfrak{c})} \omega_J(\widehat{f}) \wedge \overline{\omega_J(\widehat{f})}} \lambda_f(\mathfrak{m}) \lambda_f(\mathfrak{n})$$

where f ranges over newforms of nebentypus η^i with $\widehat{f} \in S(K_0(\mathfrak{c}))$.

Otherwise, the $\mathfrak{n}$th Fourier coefficient is zero.

Déjà vu:

Theorem (Zagier, LNM '76):

$$\begin{aligned}\langle [Z_m], \Phi \rangle_H &= c(m) E_{2,p}(z) \\ &+ r \sum_{n=1}^{\infty} \left(\sum_f \frac{\left(\int_{Z_1} \eta_{\widehat{f}} \right)^2}{(\widehat{f}, \widehat{f})} a_f(m) a_f(n) \right) q^n.\end{aligned}$$

Examples of cycles admitting classes in IH:

- Let $B_{/E}$ be a quaternion algebra. Set

$$G_B(\cdot) := \mathrm{Res}_{E/\mathbb{Q}}((B \otimes \cdot)^{\times}).$$

- Suppose that B is split by L. Then we have an injective morphism

$$\iota : G_B \hookrightarrow G = G_L.$$

- This gives, for every compact open $K \leq G(\mathbb{A}^f)$, **quaternionic Shimura subvarieties**

$$G_B(\mathbb{Q})\backslash G_B(\mathbb{A})/K_{B,\infty}(\iota^{-1}(K) \cap G_B(\mathbb{A}^f)) \hookrightarrow Y_K$$

where

$$Y_K := G(\mathbb{Q})\backslash G(\mathbb{A})/K_\infty K.$$

Examples of cycles (cont.):

We can always choose $B = M_2(E)$. Then the diagonal embedding

$$\iota : G_E \hookrightarrow G$$

gives embeddings

$$Y_{\iota^{-1}(K_0(\mathfrak{c}))\cap G_E(\mathbb{A}^f)} \hookrightarrow Y_0(\mathfrak{c})$$

and

$$X_{\iota^{-1}(K_0(\mathfrak{c}))\cap G_E(\mathbb{A}^f)} \hookrightarrow X_0(\mathfrak{c}).$$

The associated cycles intersect the cusps nontrivially.

Theorem 3 (In progress).

Let $Z \subset X_0(\mathfrak{c}_L)$ be a quaternionic Shimura subvariety associated to a quaternion algebra over E split by L. Then Z admits a canonical class

$$[Z] \in IH_{[L:\mathbb{Q}]}(X_0(\mathfrak{c})).$$

Moreover $P_{\text{new}}([Z]) \in IH^{\eta}(X_0(\mathfrak{c}))$, where

$$P_{\text{new}} : IH_{[L:\mathbb{Q}]}(X_0(\mathfrak{c})) \longrightarrow IH^{\text{new}}_{[L:\mathbb{Q}]}(X_0(\mathfrak{c}))$$

is the canonical projection, and

$$\langle [Z], [\omega(f)] \rangle_{IH} = \int_Z \omega(f).$$

Questions:

1. For good choices of cycles Z, $\langle [Z], [\omega(f)] \rangle_{IH}$ is a certain period integral. What is this period in terms of adjoint L-functions?

2. What about non-quadratic extensions?

3. What can one say about integration formulae of the type

$$\langle [Z], \omega(f) \rangle_{IH} = \int_Z \omega(f)$$

for Shimura varieties of higher $\mathbb{Q}$-rank (i.e. those with non-isolated singularities)?

(Hopefully) coming soon to a library near you:

Hilbert Modular Forms with Coefficients in Intersection Homology and Quadratic Base Change

Jayce Getz
Dept. of Math.
University of Wisconsin-Madison

Mark Goresky
Dept. of Math.
Institute for Advanced Study