Intuitionistic Necessity Revisited

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Dedicated to the memory of Hilfred Chau

Abstract

In this paper we consider an intuitionistic modal logic, which we call $\mathbf{IS4}_{\Box}$. Our approach is different to others in that we favour the natural deduction and sequent calculus proof systems rather than the axiomatic, or Hilbert-style, system. Our natural deduction formulation is simpler than other proposals. The traditional means of devising a modal logic is with reference to a model, and almost always, in terms of a Kripke model. Again our approach is different in that we favour categorical models. This facilitates not only a more abstract definition of a whole *class* of models but also a means of modelling proofs as well as provability.

1 Introduction

Prawitz-style natural deduction is a framework somewhat underestimated by modal logicians. But it is the cornerstone of functional programming, via the Curry-Howard correspondence, which is one of the most exciting applications of logic to date.

Modal logic's intensional notions of necessity and possibility have proved useful in many areas of computer science; so it would be good to extend the Curry-Howard correspondence, with all its functional programming possibilities, to modal logic. This task is nevertheless difficult in two respects. Firstly modal logics are traditionally defined in terms of classical logic, whereas functional programming corresponds to intuitionistic logic. Secondly providing any sort of formulation other than an axiomatic one is difficult for many of the proposed modal logics. Indeed providing a natural deduction formulation seems harder than providing a sequent calculus one.

Addressing the first difficulty has recently become a popular topic, with many authors trying to understand the notion of a constructive modal logic. Once the classical basis is replaced, a multitude of intuitionistic versions becomes possible and it is challenging to justify one choice over another. Most choices are made with reference to the model theory,

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although almost exclusively in terms of Kripke-style models. Kripke semantics works both for intuitionistic and modal logic, using separate accessibility relations for each, and the choices appear in deciding how these relations are to interact.

The approach we persue here is somewhat different. We also use models to guide our work but we prefer categorical ones. One reason is that, unlike the situation for Kripke semantics, we are interested in modelling not just provability but also the proofs themselves. This approach is often termed categorical proof theory, or simply categorical logic [18]. Category theory provides a language for describing abstractly what is required of a model or, more precisely, what extra structures are needed for an arbitrary category to model the logic. Checking that a candidate is a concrete model then simplifies to checking that it satisfies the abstract definition. Thus soundness, for example, need only be checked once and for all, for the abstract definition. Then all concrete models which satisfy the abstract definition are also sound. Thus categorical semantics provide a general and often simple formulation of what it is to be a model. This is of interest because it is often the case that more traditional models lack any generality or are quite complicated to describe (or both). In particular categorical semantics enable one to model some very powerful logics such as impredicative type theories and intuitionistic higher order logic.

This paper is organised as follows. In §§2–3 we give axiomatic and sequent calculus formulations of $\mathbf{IS4}_\Box$, respectively. The theorems proven in these sections are surely known to those working in this area, although we repeat them here for completeness. In §4 we give our natural deduction formulation and compare it to Prawitz's proposal for a similar logic. In §5 we define the $\lambda^\Box$-calculus, which is given by the Curry-Howard correspondence from our natural deduction formulation. We also suggest a possible computer science application for this calculus. In §6 we give in detail our categorical analysis of the necessity modality. We give a sound definition of a categorical model for $\mathbf{IS4}_\Box$.

2 An Axiomatic Formulation of $\mathbf{IS4}_\Box$

Axiomatic, or Hilbert-style, formulations are probably the more familiar method of defining modal logics. They consist of a series of axioms and a few deduction rules. For $\mathbf{IS4}_\Box$ this consists of an axiomatic presentation of intuitionistic logic augmented with three new axioms (**K**, **T** and **4**) and a new rule, *Nec.* The formulation is given in Figure 1.

It is worth explaining our axiomatic formulation. In giving the *Nec* rule it is vital to insist that there are no free assumptions, otherwise one could deduce, for example, $A \supset \Box A$. This restriction can be found in all presentations of necessity operators (e.g. [22]). Given the importance of the context for this rule, it is surprising to find that most authors disregard the context for the other rules. Here we keep the context explicit in all the rules, thus in the *Identity* rule we allow an arbitrary weakened context, *viz.* from assuming Γ, A we can deduce A. The *Axiom* rule says that from any assumptions Γ we can deduce one of the axioms from the list in Figure 1.

Where it is not obvious by context a deduction in the axiomatic system is denoted by the annotated turnstile $\vdash_A$. An important property possessed by this formulation, which is not always the case for modal logics, is the deduction theorem.

Theorem 1 *If* $\Gamma, A \vdash B$ *then there exists a proof of* $\Gamma \vdash A \supset B$.

Proof. By induction on the structure of the derivation. ■

Axioms:

$$A \supset (B \supset A)$$
$$(A \supset B \supset C) \supset ((A \supset B) \supset (A \supset C))$$
$$A \supset (B \supset A \wedge B)$$
$$A \wedge B \supset A$$
$$A \wedge B \supset B$$
$$(A \supset C) \supset ((B \supset C) \supset (A \vee B \supset C))$$
$$A \supset A \vee B$$
$$B \supset A \vee B$$
$$\bot \supset A$$

K $\Box(A \supset B) \supset (\Box A \supset \Box B)$

T $\Box A \supset A$

4 $\Box A \supset \Box\Box A$

Rules:

$$\frac{}{\Gamma, A \vdash A}\ \textit{Identity}$$

$$\frac{}{\Gamma \vdash A}\ \textit{Axiom} \text{ where } A \text{ is taken from above.}$$

$$\frac{\Gamma \vdash A \supset B \qquad \Delta \vdash A}{\Gamma, \Delta \vdash B}\ \textit{Modus Ponens}$$

$$\frac{\vdash A}{\vdash \Box A}\ \textit{Nec}$$

Figure 1: Axiomatic Formulation of $\mathbf{IS4}_\Box$.

3 A Sequent Calculus Formulation of $\mathbf{IS4}_\Box$

The sequent calculus formulation presented here is adapted from Curry's book [5] and is given in Figure 2.

Γ, Δ are used to represent sequences of formulae and A, B for single formulae. The *Exchange* rule simply allows the permutation of assumptions. The *Weakening* rule permits assumptions to be discarded and the *Contraction* rule allows an assumption to be duplicated. In what follows the *Exchange* rule is considered to be implicit, whence the convention that Γ, Δ denote *multisets*. Negation is defined, as usual for intuitionistic logic, as

$$\neg A \stackrel{\text{def}}{=} A \supset \bot.$$

The sequent calculus formulation, where we use the symbol $\vdash_S$ to represent a sequent deduction, is equivalent to the axiomatic presentation given in the previous section.

Theorem 2 *$\vdash_S \Gamma \vdash A$ iff $\vdash_A \Gamma \vdash A$.*

Proof. By induction on the structure of the derivation. For example consider the following case. Given a sequent derivation of the form

$$\frac{\begin{array}{c}\mathcal{D}\\ \Box\Gamma \vdash A\end{array}}{\Box\Gamma \vdash \Box A}\ (\Box_\mathcal{R})$$

$$\frac{}{A \vdash A}\ \textit{Axiom}$$

$$\frac{\Gamma \vdash B \qquad B, \Delta \vdash C}{\Gamma, \Delta \vdash C}\ \textit{Cut}$$

$$\frac{\Gamma, A, B, \Delta \vdash C}{\Gamma, B, A, \Delta \vdash C}\ \textit{Exchange}$$

$$\frac{}{\Gamma, \bot \vdash A}\ (\bot_{\mathcal{L}})$$

$$\frac{\Gamma \vdash C}{\Gamma, A \vdash C}\ \textit{Weakening} \qquad \frac{\Gamma, A, A \vdash C}{\Gamma, A \vdash C}\ \textit{Contraction}$$

$$\frac{\Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma, A \vee B \vdash C}\ (\vee_{\mathcal{L}}) \qquad \frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\ (\vee_{\mathcal{R}}) \quad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}\ (\vee_{\mathcal{R}})$$

$$\frac{\Gamma, A \vdash C}{\Gamma, A \wedge B \vdash C}\ (\wedge_{\mathcal{L}}) \quad \frac{\Gamma, B \vdash C}{\Gamma, A \wedge B \vdash C}\ (\wedge_{\mathcal{L}}) \qquad \frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ (\wedge_{\mathcal{R}})$$

$$\frac{\Gamma \vdash A \qquad \Gamma, B \vdash C}{\Gamma, A \supset B \vdash C}\ (\supset_{\mathcal{L}}) \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \supset B}\ (\supset_{\mathcal{R}})$$

$$\frac{\Gamma, A \vdash B}{\Gamma, \Box A \vdash B}\ (\Box_{\mathcal{L}}) \qquad \frac{\Box\Gamma \vdash A}{\Box\Gamma \vdash \Box A}\ (\Box_{\mathcal{R}})$$

Figure 2: Sequent Calculus formulation of $\mathbf{IS4}_{\Box}$.

Then by induction we have the axiomatic deduction of $\mathcal{D}$, $\vdash_A \Box\Gamma \vdash A$. Assume that $\Box\Gamma$ represents the multiset $\{\Box G_1, \ldots, \Box G_n\}$. Then we can form the following deduction.

$$\dfrac{\dfrac{\dfrac{}{\vdash \mathbf{K}} \qquad \dfrac{\vdots}{\Box G_1, \ldots, \Box G_{n-1} \vdash \Box(\Box G_n \supset A)}}{\Box G_1, \ldots \Box G_{n-1} \vdash \Box\Box G_n \supset \Box A} \qquad \dfrac{\dfrac{}{\vdash \Box G_n \supset \Box\Box G_n} \quad \dfrac{}{\Box G_n \vdash \Box G_n}}{\Box G_n \vdash \Box\Box G_n}}{\Box G_1, \ldots, \Box G_{n-1}, \Box G_n \vdash \Box A}\ M.P.$$

where the elided part is

$$\dfrac{\dfrac{\dfrac{}{\vdash \mathbf{K}} \qquad \dfrac{\dfrac{\Box\Gamma \vdash A}{\vdash \Box G_1 \supset (\Box G_2 \supset \ldots (\Box G_n \supset A))}\ D.T.^n}{\vdash \Box(\Box G_1 \supset (\Box G_2 \supset \ldots (\Box G_n \supset A)))}\ Nec}{\vdash \Box\Box G_1 \supset \Box(\Box G_2 \supset \ldots (\Box G_n \supset A))} \qquad \dfrac{\dfrac{}{\vdash \Box G_1 \supset \Box\Box G_1} \quad \dfrac{}{\Box G_1 \vdash \Box G_1}}{\Box G_1 \vdash \Box\Box G_1}}{\Box G_1 \vdash \Box(\Box G_2 \supset \ldots (\Box G_n \supset A))}$$

where $D.T.^n$ represents n applications of the Deduction Theorem and $\mathbf{K}$ denotes a suitable instance of the $\mathbf{K}$ axiom from Figure 1. ■

An important property of sequent formulations is the so-called cut-elimination theorem. Here instances of the *Cut* rule are analysed and replaced with instances on smaller proofs (the technical details are a little delicate; Gallier [8] gives a nice explanation). The important new case for our logic is an instance of a $(\Box_{\mathcal{R}}, \Box_{\mathcal{L}})$-cut, *viz.*

$$\dfrac{\dfrac{\Box\Gamma \vdash A}{\Box\Gamma \vdash \Box A}\,(\Box_{\mathcal{R}}) \qquad \dfrac{\Delta, A \vdash B}{\Delta, \Box A \vdash B}\,(\Box_{\mathcal{L}})}{\Box\Gamma, \Delta \vdash B}\,Cut$$

which is rewritten to

$$\dfrac{\Box\Gamma \vdash A \qquad \Delta, A \vdash B}{\Box\Gamma, \Delta \vdash B}\,Cut.$$

Theorem 3 *Given a derivation π of $\Gamma \vdash A$, a derivation π' of $\Gamma \vdash A$ can be found which contains no instances of the* Cut *rule.*

4 A Natural Deduction Formulation of $\mathbf{IS4}_{\Box}$

In a natural deduction system, originally due to Gentzen [9], but subsequently expounded by Prawitz [19], a deduction is a derivation of a proposition from a finite set of assumption packets, using some predefined set of inference rules. Within a deduction, we may 'discharge' any number of assumption packets. Assumption packets can be given natural number labels (denoted by x) and applications of inference rules can be annotated with the labels of those packets which they discharge.

The formulation is given in Figure 3. Our formulation differs from others in its simpler treatment of the modality.

Some care should be taken with the $(\Box_{\mathcal{I}})$ rule. The semantic braces, $[\![\cdots]\!]$, mean not only that *all* the assumptions are modal[1] but they are *all* discharged (and re-introduced). The advantage of this formulation of this rule is that it satisfies a fundamental feature of natural deduction in that it is *closed under substitution.* One might have been tempted to give the rule for $(\Box_{\mathcal{I}})$ as

$$\begin{array}{c} \Box A_1 \cdots \Box A_k \\ \vdots \\ \dfrac{B}{\Box B}\,(\Box_{\mathcal{I}}), \end{array}$$

where the assumptions must all be modal but are not discharged and reintroduced, though clearly this rule is *not* closed under substitution. For example, substituting for $\Box A_1$, the deduction

$$\dfrac{C \supset \Box A_1 \qquad C}{\Box A_1}\,(\supset_{\mathcal{E}})$$

we get the following deduction

[1] In comparison with our $(\supset_{\mathcal{I}})$ rule where the standard notation is taken to mean that only one assumption, A, is discharged.

$$\frac{\begin{matrix}\vdots\\ \bot\end{matrix}}{A}\,(\bot_{\mathcal{E}})$$

$$\frac{\begin{matrix}[A^x]\\ \vdots\\ B\end{matrix}}{A \supset B}\,(\supset_{\mathcal{I}})_x \qquad \frac{\begin{matrix}\vdots & & \vdots\\ A \supset B & & A\end{matrix}}{B}\,(\supset_{\mathcal{E}})$$

$$\frac{\begin{matrix}\vdots & \vdots\\ A & B\end{matrix}}{A \wedge B}\,(\wedge_{\mathcal{I}}) \qquad \frac{\begin{matrix}\vdots\\ A \wedge B\end{matrix}}{A}\,(\wedge_{\mathcal{E}}) \quad \frac{\begin{matrix}\vdots\\ A \wedge B\end{matrix}}{B}\,(\wedge_{\mathcal{E}})$$

$$\frac{\begin{matrix}\vdots\\ A\end{matrix}}{A \vee B}\,(\vee_{\mathcal{I}}) \quad \frac{\begin{matrix}\vdots\\ B\end{matrix}}{A \vee B}\,(\vee_{\mathcal{I}}) \qquad \frac{\begin{matrix} & [A] & [B]\\ \vdots & \vdots & \vdots\\ A \vee B & C & C\end{matrix}}{C}\,(\vee_{\mathcal{E}})$$

$$\frac{\begin{matrix}\vdots\\ \Box B\end{matrix}}{B}\,(\Box_{\mathcal{E}}) \qquad \frac{\begin{matrix} & & & [\![\Box A_1^{x_1} \cdots \Box A_k^{x_k}]\!]\\ \vdots & & \vdots & \vdots\\ \Box A_1 & \ldots & \Box A_k & B\end{matrix}}{\Box B}\,(\Box_{\mathcal{I}})_{x_1,\ldots,x_k}$$

Figure 3: Natural Deduction Formulation of $\mathbf{IS4}_\Box$.

$$\frac{\begin{matrix}\dfrac{C \supset \Box A_1 \qquad C}{\Box A_1}\,(\supset_{\mathcal{E}}) \quad \cdots \Box A_k\\ \vdots\\ B\end{matrix}}{\Box B}\,(\Box_{\mathcal{I}})$$

which is no longer a valid deduction as the assumptions are not all modal. We conclude that $(\Box_{\mathcal{I}})$ should be formulated as in Figure 3, where the substitutions are given explicitly.[2]

It is possible to present natural deduction rules in a 'sequent-style', where given a sequent $\Gamma \vdash A$, then Γ represents all the undischarged assumptions and A represents the conclusion of the deduction. This formulation should not be confused with the sequent calculus formulation, which differs by having operations which act on the left and right of the turnstile, rather than rules for the introduction and elimination of logical operators. The 'sequent-style' formulation of natural deduction for $\mathbf{IS4}_\Box$ is given in Figure 4.

Two important admissible rules are

$$\frac{\Gamma \vdash B}{\Gamma, A \vdash B}\,\textit{Weakening} \qquad \text{and} \qquad \frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B}\,\textit{Contraction.}$$

[2]This rule originates from the natural deduction formulation of intuitionistic linear logic [3].

$$\Gamma, A \boldsymbol{\vdash} A$$

$$\frac{\Gamma \boldsymbol{\vdash} \bot}{\Gamma \boldsymbol{\vdash} A}\,(\bot_{\mathcal{E}})$$

$$\frac{\Gamma, A \boldsymbol{\vdash} B}{\Gamma \boldsymbol{\vdash} A \supset B}\,(\supset_{\mathcal{I}}) \qquad \frac{\Gamma \boldsymbol{\vdash} A \supset B \qquad \Gamma \boldsymbol{\vdash} A}{\Gamma \boldsymbol{\vdash} B}\,(\supset_{\mathcal{E}})$$

$$\frac{\Gamma \boldsymbol{\vdash} A \qquad \Gamma \boldsymbol{\vdash} B}{\Gamma \boldsymbol{\vdash} A \wedge B}\,(\wedge_{\mathcal{I}}) \quad \frac{\Gamma \boldsymbol{\vdash} A \wedge B}{\Gamma \boldsymbol{\vdash} A}\,(\wedge_{\mathcal{E}}) \quad \frac{\Gamma \boldsymbol{\vdash} A \wedge B}{\Gamma \boldsymbol{\vdash} B}\,(\wedge_{\mathcal{E}})$$

$$\frac{\Gamma \boldsymbol{\vdash} A}{\Gamma \boldsymbol{\vdash} A \vee B}\,(\vee_{\mathcal{I}}) \quad \frac{\Gamma \boldsymbol{\vdash} B}{\Gamma \boldsymbol{\vdash} A \vee B}\,(\vee_{\mathcal{I}})$$

$$\frac{\Gamma \boldsymbol{\vdash} A \vee B \qquad \Gamma, A \boldsymbol{\vdash} C \qquad \Gamma, B \boldsymbol{\vdash} C}{\Gamma \boldsymbol{\vdash} C}\,(\vee_{\mathcal{E}})$$

$$\frac{\Gamma \boldsymbol{\vdash} \Box A_1 \quad \cdots \quad \Gamma \boldsymbol{\vdash} \Box A_k \qquad \Box A_1, \ldots, \Box A_k \boldsymbol{\vdash} B}{\Gamma \boldsymbol{\vdash} \Box B}\,(\Box_{\mathcal{I}})$$

$$\frac{\Gamma \boldsymbol{\vdash} \Box A}{\Gamma \boldsymbol{\vdash} A}\,(\Box_{\mathcal{E}})$$

Figure 4: Natural Deduction formulation of $\mathbf{IS4}_{\Box}$ in sequent-style.

This formulation (where we use the symbol $\vdash_N$ to represent a natural deduction) is equivalent to the axiomatic formulation given in §2.

Theorem 4 $\vdash_N \Gamma \boldsymbol{\vdash} A$ *iff* $\vdash_A \Gamma \boldsymbol{\vdash} A$.

Proof. By induction on the structure of the derivation. ∎

4.1 Comparison with Prawitz's Proposal

In his monograph [19, Chapter VI] Prawitz considers adding both necessity and possibility operators to natural deduction formulations of both intuitionistic and classical logic. Our system is equivalent in terms of provability to the system he calls I_{S4}. Prawitz also noticed the problem of closure under substitution, but his solution involves a new notion of "essentially modal" formulae. What this amounts to is a relaxing of the restriction that all the undischarged formulae are modal, but rather that there is somewhere in the deduction a *complete set* of modal formulae which could have had deductions substituted in for them. In tree-form this amounts to the rule (where the complete set of formulae is in bold face)

$$\begin{array}{ccc} \Delta_1 & & \Delta_k \\ \vdots & & \vdots \\ \Box\mathbf{A_1} & \cdots & \Box\mathbf{A_k} \\ & \vdots & \\ & B & \\ \end{array}$$

$$\frac{B}{\Box B}\ (\Box_{\mathcal{I}}).$$

Of course there is the extra work of finding this complete set; and indeed there may be more than one (the rather serious proof and model theoretic consequences of this are discussed in §7). We feel that our proposal is conceptually clearer: the only feature we use is the discharging of formulae, which is already present.

4.2 Normalisation

With a natural deduction formulation we can produce so-called detours in a deduction, which arise where we introduce a logical connective only to eliminate it immediately afterwards. We can define a reduction relation, denoted $\leadsto_\beta$, (and called β-reduction) by considering each case in turn. The treatment of the familiar intuitionistic connectives is entirely standard and the reader is referred to other works [19]. The new case is where $(\Box_{\mathcal{I}})$ is followed by $(\Box_{\mathcal{E}})$. Thus

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Box A_1\end{array} \quad \ldots \quad \begin{array}{c}\vdots\\ \Box A_k\end{array} \qquad \begin{array}{c}[\![\Box A_1 \ldots \Box A_k]\!]\\ \vdots\\ B\end{array}}{\Box B}\,(\Box_{\mathcal{I}})}{B}\,(\Box_{\mathcal{E}})$$

is reduced to

$$\begin{array}{ccc} \vdots & & \vdots \\ [\![\Box A_1 & \ldots & \Box A_k]\!] \\ & \vdots & \\ & B. & \end{array}$$

As is standard, we say that a proof containing no instances of a β-reduction is in *β-normal form.* Our formulation of $\mathbf{IS4}_\Box$ has the following property.

Proposition 1 *If $\Gamma \vdash A$ in $\mathbf{IS4}_\Box$ then there is a natural deduction of A from Γ which is in β-*normal form.

5 Term Assignment for $\mathbf{IS4}_\Box$

The Curry-Howard correspondence [11] relates constructive logics to typed λ-calculi. It essentially annotates each stage of a deduction with a 'term', which is an encoding of the construction of the deduction so far. Consequently a logic can be viewed as a type system for a term assignment system. The correspondence also links proof normalisation to term reduction.

The Curry-Howard correspondence can be applied to the natural deduction formulation to obtain the term assignment system given in Figure 5. It should be pointed out that the natural number labels mentioned above, are replaced by (the more familiar) variable names. The resulting calculus we call the $\lambda^{\Box}$-calculus.

$$x\colon A \rhd x\colon A$$

$$\frac{\Gamma \rhd M\colon \bot}{\Gamma \rhd \nabla_A(M)\colon A}\ (\bot_{\mathcal{E}})$$

$$\frac{\Gamma, x\colon A \rhd M\colon B}{\Gamma \rhd \lambda x\colon A.M\colon A \to B}\ (\to_{\mathcal{I}}) \qquad \frac{\Gamma \rhd M\colon A \to B \qquad \Gamma \rhd N\colon A}{\Gamma \rhd MN\colon B}\ (\to_{\mathcal{E}})$$

$$\frac{\Gamma \rhd M\colon A \qquad \Gamma \rhd N\colon B}{\Gamma \rhd \langle M, N\rangle\colon A \times B}\ (\times_{\mathcal{I}}) \quad \frac{\Gamma \rhd M\colon A \times B}{\Gamma \rhd \mathsf{fst}(M)\colon A}\ (\times_{\mathcal{E}}) \quad \frac{\Gamma \rhd M\colon A \times B}{\Gamma \rhd \mathsf{snd}(M)\colon B}\ (\times_{\mathcal{E}})$$

$$\frac{\Gamma \rhd M\colon A}{\Gamma \rhd \mathsf{inl}(M)\colon A + B}\ (+_{\mathcal{I}}) \qquad \frac{\Gamma \rhd M\colon B}{\Gamma \rhd \mathsf{inr}(M)\colon A + B}\ (+_{\mathcal{I}})$$

$$\frac{\Gamma \rhd M\colon A + B \qquad \Gamma, x\colon A \rhd N\colon C \qquad \Gamma, y\colon B \rhd P\colon C}{\Gamma \rhd \mathsf{case}\ M\ \mathsf{of}\ \mathsf{inl}(x)\ \to\ N \parallel \mathsf{inr}(y)\ \to\ P\colon C}\ (+_{\mathcal{E}})$$

$$\frac{\Gamma \rhd M_1\colon \Box A_1 \ \cdots\ \Gamma \rhd M_k\colon \Box A_k \qquad x_1\colon \Box A_1, \ldots, x_k\colon \Box A_k \rhd N\colon B}{\Gamma \rhd \mathsf{box}\ N\ \mathsf{with}\ M_1, \ldots, M_k\ \mathsf{for}\ x_1, \ldots, x_k\colon \Box B}\ (\Box_{\mathcal{I}})$$

$$\frac{\Gamma \rhd M\colon \Box A}{\Gamma \rhd \mathsf{unbox}(M)\colon A}\ (\Box_{\mathcal{E}})$$

Figure 5: Term Assignment for $\mathbf{IS4}_{\Box}$

An important property of our system is that substitution is well-defined in the following sense.

Theorem 5 *If* $\Gamma \rhd N\colon A$ *and* $\Gamma, x\colon A \rhd M\colon B$ *then* $\Gamma \rhd M[x := N]\colon B$.

Proof. By induction on the derivation $\Gamma, x\colon A \rhd M\colon B$. ∎

Before we continue, a quick word concerning the $(\Box_{\mathcal{I}})$ rule. At first sight this seems to imply an ordering of the M_i and x_i subterms. However, the *Exchange* rule (which does not introduce any additional syntax) tells us that any such order is really just the effect of writing terms in a sequential manner on the page.

The reduction rules derived from §4.2 can be given at the level of terms. These are given in Figure 6 where the symbol $\leadsto_\beta$ is used to denote term reduction. We have also used the shorthand $\vec{M}$ in place of the sequence $M_1, \ldots M_k$. The last reduction rule corresponds to the proof reduction discussed in §4.2.

$$
\begin{array}{lll}
(\lambda x{:}\, A.M)N & \leadsto_\beta & M[x := N] \\
\mathsf{fst}(\langle M, N\rangle) & \leadsto_\beta & M \\
\mathsf{snd}(\langle M, N\rangle) & \leadsto_\beta & N \\
\mathsf{case}\ \mathsf{inl}(M)\ \mathsf{of}\ \mathsf{inl}(x) \rightarrow N \parallel \mathsf{inr}(y) \rightarrow P & \leadsto_\beta & N[x := M] \\
\mathsf{case}\ \mathsf{inr}(M)\ \mathsf{of}\ \mathsf{inl}(x) \rightarrow N \parallel \mathsf{inr}(y) \rightarrow P & \leadsto_\beta & P[y := M] \\
\mathsf{unbox}(\mathsf{box}\ N\ \mathsf{with}\ \vec{M}\ \mathsf{for}\ \vec{x}) & \leadsto_\beta & N[\vec{x} := \vec{M}]
\end{array}
$$

Figure 6: β-reduction rules.

5.1 A Computational Interpretation

As is now well known, the typed λ-calculus can be thought of as a prototypical functional programming language. An alternative view is that it can be thought of as an intermediate language inside a functional language compiler. (The classic treatment of this is in Peyton Jones' book [12].) The equational reasoning of the λ-calculus enables one to view compiler optimisations as manipulations of terms of the intermediate language.

Inside a compiler there is a difference between values stored directly in the local stack and those stored in the heap. Of course in the intermediate language (the λ-calculus) such operational differences are not distinguished. Certain optimisations in compilers involve moving between these different representations.

It seems that the $\lambda^\Box$-calculus is an appropriate language for such distinctions to be made explicit at the term, and type, level. Thus a value of type A is to be considered a 'local' value of (type A) and a value of type $\Box A$ a stored one. The restriction of the $\Box_\mathcal{R}$ rule can be interpreted as follows: if a value is to be placed on the heap then it must only reference values also on the heap (i.e. the free variables should be of type $\Box B$). Manipulations of values to and from the heap are now represented by explicit terms. This is analogous to Moggi's [16] proposal of differentiating, at the term level, between canonical values and computations.[3] Indeed it would appear that a language combining both Moggi's ideas and those above, is worthy of further study.[4]

6 The Categorical Model

The fundamental idea of a categorical treatment of proof theory is that propositions should be interpreted as the objects of a category and proofs should be interpreted as morphisms. The proof rules correspond to natural transformations between appropriate hom-functors. The proof theory gives a number of reduction rules, which can be viewed as equalities between proofs. In particular these equalities should hold in the categorical model.

Other categorical studies have been carried out, notably by Flagg [7]; Meloni and Ghilardi [10] and Reyes and Zolfaghari [20]. However these have been mainly concerned

[3] Moggi's language, the *computational λ-calculus*, can also be seen as a modal logic [2].

[4] In fact, this idea is being studied (and considerably extended) by P.N. Benton (private communication).

with categorical *model* theory, rather than categorical *proof* theory. In particular, they all assume an isomorphism, $\Box\Box A \cong \Box A$. In this work we have morphisms in both directions (as they are provably equivalent) but we have *not* collapsed the model so that they are isomorphic.

Let us fix some notation. The interpretation of a proof is represented using semantic braces, $[\![-]\!]$, making the usual simplification of using the same letter to represent a proposition as its interpretation. Given a term $\Gamma \triangleright M\colon A$ where $M \rightsquigarrow_\beta N$, we shall write $\Gamma \triangleright M = N\colon A$.

Definition 1 *A category, $\mathcal{C}$, is said to be a categorical model of a logic/term calculus iff*

1. *For all proofs $\Gamma \triangleright M\colon A$ there is a morphism $[\![M]\!]\colon \Gamma \to A$ in $\mathcal{C}$; and*
2. *For all proof equalities $\Gamma \triangleright M = N\colon A$ it is the case that $[\![M]\!] =_{\mathcal{C}} [\![N]\!]$ (where $=_{\mathcal{C}}$ represents equality of morphisms in the category $\mathcal{C}$).*

Given this definition we simply analyse the introduction and elimination rules for each connective. Both this and consideration of the reduction rules should suggest a particular categorical structure to model the connective. The case for intuitionistic logic is well known; the reader is referred to Lambek and Scott's book [13] for a good discussion. Essentially the categorical model of intuitionistic logic (with disjunction) is a cartesian closed category (CCC) with coproducts. Hence all we need do here is consider the modality, which we shall do in some detail. The less-categorically minded reader may wish simply to skip to Definition 2.

The introduction rule for the modality is of the form

$$\frac{\Gamma \triangleright M_1\colon \Box A_1 \quad \cdots \quad \Gamma \triangleright M_k\colon \Box A_k \qquad x_1\colon \Box A_1, \ldots, x_k\colon \Box A_k \triangleright N\colon B}{\Gamma \triangleright \mathsf{box}\ N\ \mathsf{with}\ \vec{M}\ \mathsf{for}\ \vec{x}\colon \Box B}\ (\Box_{\mathcal{I}})$$

To interpret this rule we need a natural transformation with components

$$\Phi_\Gamma\colon \mathcal{C}(\Gamma, \Box A_1) \times \cdots \times \mathcal{C}(\Gamma, \Box A_k) \times \mathcal{C}(\Box A_1 \times \cdots \times \Box A_k, B) \to \mathcal{C}(\Gamma, \Box B)$$

Given morphisms $e_i\colon \Gamma \to \Box A_i$, $c\colon \Gamma' \to \Gamma$ and $d\colon \Box A_1 \times \cdots \times \Box A_k \to B$, naturality gives the equation

$$c; \Phi_\Gamma(e_1, \ldots, e_k, d) \quad = \quad \Phi_{\Gamma'}((c; e_1), \ldots, (c; e_k), d).$$

In particular if we have morphisms $m_i\colon \Gamma \to \Box A_i$ then we take $c = \langle m_1, \ldots, m_k \rangle$, e_i to be the i-th product projection, written π_i, and d to be some morphism $p\colon \Box A_1 \times \cdots \times \Box A_k \to B$, then by naturality we have

$$\langle m_1, \ldots, m_k \rangle; \Phi_{\Box A_1, \ldots, \Box A_k}(\pi_1, \ldots, \pi_k, p) = \Phi_{\Box A_1, \ldots, \Box A_k}(m_1, \ldots, m_k, p).$$

Thus $\Phi(m_1, \ldots, m_k, p)$ can be expressed as the composition $\langle m_1, \ldots, m_k \rangle; \Psi(p)$, where Ψ is a transformation

$$\Psi\colon \mathcal{C}(\Box A_1 \times \cdots \times \Box A_k, B) \to \mathcal{C}(\Box A_1 \times \cdots \times \Box A_k, \Box B).$$

For the moment, the effect of this transformation will be written as $(-)^*$ and so we can make the preliminary definition

$$\llbracket \Gamma \triangleright \mathsf{box}\ N\ \mathsf{with}\ M_1, \ldots, M_k\ \mathsf{for}\ x_1, \ldots, x_k : \Box B \rrbracket \stackrel{\mathrm{def}}{=}$$
$$\langle (\llbracket \Gamma \triangleright M_1 : \Box A_1 \rrbracket), \ldots, (\llbracket \Gamma \triangleright M_k : \Box A_k \rrbracket) \rangle ; (\llbracket x_1 : \Box A_1, \ldots, x_k : \Box A_k \triangleright N : B \rrbracket)^*$$

The elimination rule for the modality is of the form

$$\frac{\Gamma \triangleright M : \Box A}{\Gamma \triangleright \mathsf{unbox}(M) : A}\ (\Box_{\mathcal{E}})$$

To interpret this rule we need a natural transformation

$$\Phi : \mathcal{C}(-, \Box A) \to \mathcal{C}(-, A).$$

It follows from the Yoneda Lemma [14, Page 61] that there is the bijection

$$[\mathcal{C}^{op}, \mathbf{Sets}](\mathcal{C}(-, \Box A), \mathcal{C}(-, A)) \cong \mathcal{C}(\Box A, A).$$

By constructing this isomorphism one can see that the components of Φ are induced by postcomposition by a morphism $\varepsilon : \Box A \to A$. Thus we make the definition

$$\llbracket \Gamma \triangleright \mathsf{unbox}(M) : A \rrbracket \stackrel{\mathrm{def}}{=} \llbracket \Gamma \triangleright M : \Box A \rrbracket ; \varepsilon.$$

From Figure 6 we have the term equality

$$\frac{\Gamma \triangleright M_1 : \Box A_1 \quad \cdots \quad \Gamma \triangleright M_k : \Box A_k \qquad x_1 : \Box A_1, \ldots, x_k : \Box A_k \triangleright N : B}{\Gamma \triangleright \mathsf{unbox}(\mathsf{box}\ N\ \mathsf{with}\ \vec{x}\ \mathsf{for}\ \vec{M}) = N[\vec{x} := \vec{M}] : B}$$

Taking morphisms $m_i : \Gamma \to \Box A_i$ and $p : \Box A_1 \times \cdots \times \Box A_k \to B$, say, this term equality amounts to the categorical equality

$$\langle m_1, \ldots, m_k \rangle ; (p)^* ; \varepsilon \quad = \quad \langle m_1, \ldots, m_k \rangle ; p. \tag{1}$$

We can certainly define an operation

$$\begin{array}{l} \Box : \mathcal{C}(\Gamma, A) \to \mathcal{C}(\Box\Gamma, \Box A), \\ f \mapsto (\varepsilon ; f)^*. \end{array}$$

We shall make the simplifying assumption that this operation is a *functor*. However, notice that if Γ is the object $A_1 \times \cdots \times A_k$, then $\Box\Gamma$ will be represented by $\Box(A_1 \times \cdots \times A_k)$, but clearly we mean $\Box A_1 \times \cdots \times \Box A_k$. Thus we shall make the further simplifying assumption that $\Box$ is a *symmetric monoidal functor*, $(\Box, \mathsf{m}_{A,B}, \mathsf{m}_1)$. This notion is originally due to Eilenberg and Kelly [6]. In essence this provides a natural transformation

$$\mathsf{m}_{A,B} : \Box A \times \Box B \to \Box(A \times B)$$

and morphism

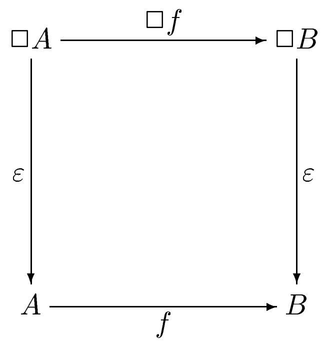
$$\mathsf{m}_1 : 1 \to \Box 1$$

which satisfy a number of conditions which are detailed in Appendix A.

Equation 1 gives

$$(\varepsilon_A; f)^*; \varepsilon_B = \varepsilon_A; f$$

for any morphism $f: A \to B$; or, in other words the diagram



commutes. Given the assumption that $\Box$ is a symmetric monoidal functor, this diagram suggests that ε is a monoidal natural transformation. Again the unfamiliar reader is referred to the appendix for definitions.

We have that from the identity morphism $\mathsf{id}_{\Box A}: \Box A \to \Box A$, we can form the canonical morphism $\delta_A \stackrel{\text{def}}{=} (\mathsf{id}_{\Box A})^*$. Equation 1 gives

$$\delta_A; \varepsilon_{\Box A} = \mathsf{id}_{\Box A}.$$

The categorically-minded reader will recognise this equation as one of the three for a *comonad*. We shall make the simplifying assumption that not only does $(\Box, \varepsilon, \delta)$ form a comonad but that δ is also a monoidal natural transformation. Hence the comonad is actually a *monoidal* comonad. Thus our definition of a categorical model for $\mathbf{IS4}_\Box$ is as follows.

Definition 2 *A categorical model for* $\mathbf{IS4}_\Box$ *consists of a cartesian closed category with coproducts, together with a monoidal comonad* $(\Box, \varepsilon, \delta, \mathsf{m}_{A,B}, \mathsf{m}_1)$.

We can now finalise the interpretation of the introduction rule for the modality.

$$\begin{array}{rcl} [\![\Gamma \rhd \mathsf{box}\ N\ \mathsf{with}\ \vec{M}\ \mathsf{for}\ \vec{x}: \Box B]\!] & \stackrel{\text{def}}{=} & \langle [\![\Gamma \rhd M_1: \Box A_1]\!], \ldots, [\![\Gamma \rhd M_k: \Box A_k]\!] \rangle; \\ & & \delta_{A_1} \times \cdots \times \delta_{A_k}; \mathsf{m}_{\Box A_1, \ldots, \Box A_k}; \\ & & \Box [\![x_1: \Box A_1, \ldots, x_k: \Box A_k \rhd N: B]\!] \end{array}$$

Fact. Recall that by condition 2 of our definition of a categorical model (Definition 1) if two proofs are equal then so are their denotations. In more traditional model-theory parlance this is a *soundness* theorem. Hence any concrete model satisfying the abstract conditions of Definition 2 is a sound model of $\mathbf{IS4}_\Box$.

7 Prawitz's Formulation and the Categorical Model

Although Prawitz's formulation has the appearence of being equivalent to the formulation presented in this paper, in fact it has rather unfortunate proof and model theoretic consequences. Consider the following deduction in Prawitz's formulation.

$$\dfrac{\dfrac{\dfrac{\dfrac{\Box\Box\Box A}{(1)\ \ \Box\Box A}\,(\Box_{\mathcal{E}})}{(2)\ \ \Box A}\,(\Box_{\mathcal{E}})}{A}\,(\Box_{\mathcal{E}})}{\Box A}\,(\Box_{\mathcal{I}})$$

The problem is deciding which formula was the (modal) assumption when the $\Box$ was introduced (the so-called 'complete set' from §4.1). In particular two possibilities are (1) and (2). In our formulation presented in §4, these alternatives represent two distinct derivations, *viz.*

$$\dfrac{\dfrac{\Box\Box\Box A}{\Box\Box A}\,(\Box_{\mathcal{E}}) \qquad\qquad \dfrac{\dfrac{[\![\Box\Box A]\!]}{\Box A}\,(\Box_{\mathcal{E}})}{A}\,(\Box_{\mathcal{E}})}{\Box A}\,(\Box_{\mathcal{I}})$$

and

$$\dfrac{\dfrac{\dfrac{\Box\Box\Box A}{\Box\Box A}\,(\Box_{\mathcal{E}})}{\Box A}\,(\Box_{\mathcal{E}}) \qquad\qquad \dfrac{[\![\Box A]\!]}{A}\,(\Box_{\mathcal{E}})}{\Box A}\,(\Box_{\mathcal{I}})$$

Prawitz's formulation essentially collapses these two derivations into one. In other words his formulation forces a seemingly unnecessary identification of proofs. Let us consider the consequences of this identification with respect to the categorical model. The two derivations above are modelled by the morphisms

$$\varepsilon_{\Box\Box A}; \delta_{\Box A}; \Box(\varepsilon_{\Box\Box A}); \Box(\varepsilon_{\Box A}) \colon \Box\Box\Box A \to A$$

and

$$\varepsilon_{\Box\Box A}; \varepsilon_{\Box A}; \delta_{A}; \Box\varepsilon_{A} \colon \Box\Box\Box A \to A$$

respectively. Insisting on these being equal amounts to the equality

$$\varepsilon_{\Box\Box A}; \Box\varepsilon_{A} = \varepsilon_{\Box\Box A}; \varepsilon_{\Box A}.$$

Precomposing this equality with the morphism $\delta_{\Box A}$ gives

$$\Box\varepsilon_{A} = \varepsilon_{\Box A}.$$

It is easy to see that this is sufficient to make the comonad *idempotent*, i.e. $\Box A \cong \Box\Box A$. It is worth reiterating that our formulation does *not* impose this identification of proofs and consequently does not force an idempotency.

8 Conclusions

In this paper we have considered the propositional, intuitionistic modal logic $\mathbf{IS4}_{\Box}$, and have given axiomatic, sequent calculus and natural deduction formulations; the corresponding term assignment system as well as a general categorical model.

As menioned in the introduction we place particular importance on the natural deduction proof system. In his seminal monograph, Prawitz also considered formulations of modal operators although he requires extra machinery specifically for these modalities. At the level of proofs his formulation introduces seemingly unnecessary identifications, which in the model forces an idempotency. Other authors have proposed alternative natural deduction formulations but again they all require significant extensions to the essential nature of natural deduction (for example, by indexing formulae with certain information). Examples of other proposals are those of Segerberg [4, pages 29–30], Benevides and Maibaum [1] and Mints [15, Pages 221–294].[5] Again we reiterate the conceptual simplicity of our proposal.

We also prefer the use of categorical models. Unlike other categorical work we have placed emphasis on modelling the proof theory not just provability. Our resulting model is considerably simpler than other proposals.

For the future we should like to consider other modal logics within our framework. It is clear that not all of the hundreds of modal logics will fit into our framework. However we do not view this as a weakness of our work. Rather we feel it is important to identify those modal logics which have interesting proof theories and mathematically appealing classes of models. We should also like to pursue the computational interpretation discussed in §5.1.

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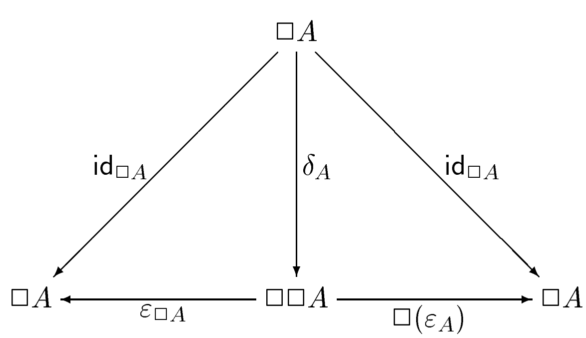
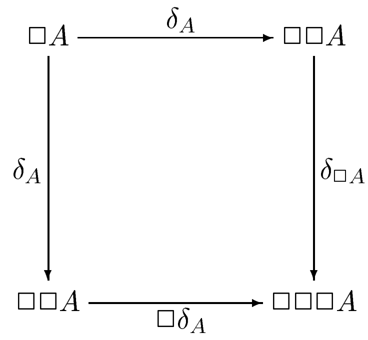
A Monoidal Comonads

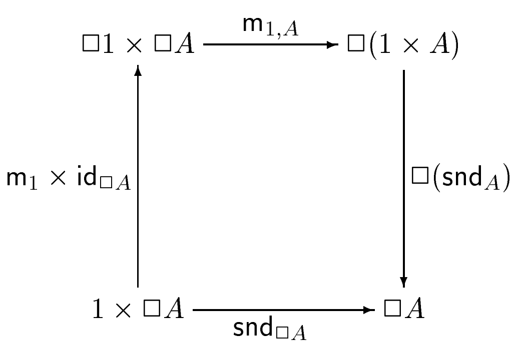
In this appendix we simply spell out the conditions implied by requiring that $(\Box, \delta, \varepsilon, \mathsf{m}_{A,B}, \mathsf{m}_1)$ is a monoidal comonad. These notions are due to Eilenberg and Kelly [6].

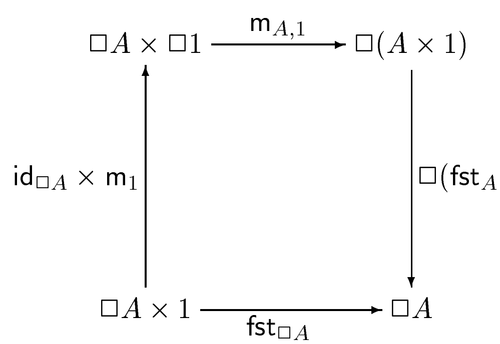
Firstly requiring that $(\Box, \varepsilon, \delta)$ form a comonad amounts to the following two diagrams.

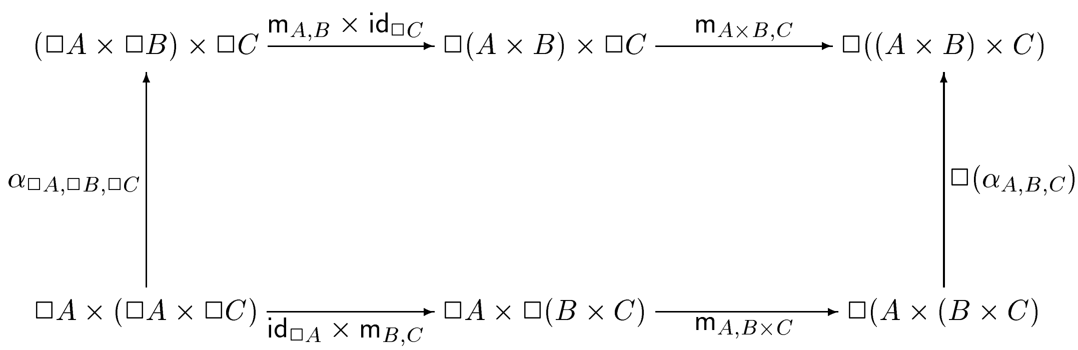


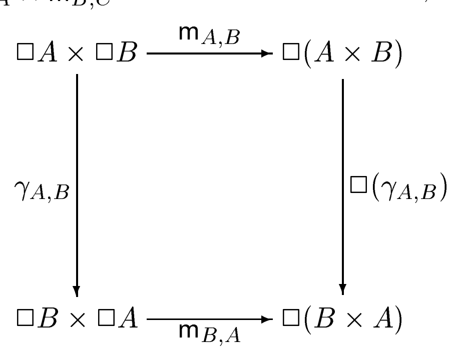


Requiring that $(\Box, \mathsf{m}_{A,B}, \mathsf{m}_1)$ is a monoidal functor amounts to the following four commuting diagrams.









Requiring that ε is a monoidal natural transformation amounts to the following two commuting diagrams.

$$\begin{array}{ccc} \Box A \times \Box B & \xrightarrow{\mathsf{m}_{A,B}} & \Box(A \times B) \\ & \searrow{\scriptstyle \varepsilon_A \times \varepsilon_B} & \downarrow{\scriptstyle \varepsilon_{A \times B}} \\ & & A \times B \end{array} \qquad \begin{array}{ccc} 1 & & \\ \downarrow{\scriptstyle \mathsf{m}_1} & \searrow{\scriptstyle \mathsf{id}_1} & \\ \Box 1 & \xrightarrow[\varepsilon_1]{} & 1 \end{array}$$

Requiring that δ is a monoidal natural transformation amounts to the following two commuting diagrams.

$$\begin{array}{ccccc} \Box A \times \Box B & & \xrightarrow{\mathsf{m}_{A,B}} & & \Box(A \times B) \\ \downarrow{\scriptstyle \delta_A \times \delta_B} & & & & \downarrow{\scriptstyle \delta_{A \times B}} \\ \Box\Box A \times \Box\Box B & \xrightarrow[\mathsf{m}_{\Box A, \Box B}]{} & \Box(\Box A \times \Box B) & \xrightarrow[\Box(\mathsf{m}_{A,B})]{} & \Box\Box(A \times B) \end{array}$$

$$\begin{array}{ccc} 1 & \xrightarrow{\mathsf{m}_1} & \Box 1 \\ \downarrow{\scriptstyle \mathsf{m}_1} & & \downarrow{\scriptstyle \delta_1} \\ \Box 1 & \xrightarrow[\Box \mathsf{m}_1]{} & \Box\Box 1 \end{array}$$