# SIMUW Theory of Equations: Fields

A field is a set $F$ with two operations, $+$ and $\cdot$, that satisfy the following list of properties. Each of these operations takes two elements of $F$ as input and returns an element of $F$. Typically, one writes $ab$ instead of $a \cdot b$.

1. Associativity: for all $x, y, z \in F$, $(x + y) + z = x + (y + z)$ and $(xy)z = x(yz)$.
2. Commutativity: for all $x, y \in F$, $x + y = y + x$ and $xy = yx$.
3. Identity elements: there are unique elements 0 and 1 in $F$ such that $0 + x = x$ and $1 \cdot x = x$ for all $x \in F$. We also require $0 \neq 1$.
4. Inverse elements: each element $x \in F$ has a unique additive inverse $-x \in F$ such that $-x + x = 0$, and each nonzero element $x \in F$ has a unique multiplicative inverse $x^{-1} \in F$ such that $x^{-1} \cdot x = 1$.
5. Distributivity: for all $x, y, z \in F$, $x(y + z) = xy + xz$.

These properties should all be familiar from studying the real numbers. One way to think of a field is as a set in which enough rules hold that one can do algebra (but of course this is a little too vague).

A field has characteristic $p$ if

$$\overbrace{1 + 1 + \cdots + 1}^{p \text{ times}} = 0,$$

and characteristic 0 otherwise (i.e., adding ones never yields zero). One can check that if there is a field of characteristic $p > 0$ then $p$ must be prime.

I've written the definition above using the symbols $+$, $\cdot$, 0, and 1 to make it look familiar, but that may be misleading. There's no reason the operations and identity elements in a field should be the usual ones. Let's use the symbols $\oplus$, $\otimes$, $Z$ (for "zero"), and $I$ (for "identity") when we want to deal with weird operations, so they don't get confused with the usual ones. Also, notice that the inverse elements do not need to be the usual ones, when the operations are weird.

The order of operations in a field is the same as usual. In other words, do multiplications before additions unless parentheses intervene.

Which of the following are fields? (Give full explanations.)

1. $\mathbb{R}$, with the usual $+$ and $\cdot$.
2. $\mathbb{C}$, with the usual $+$ and $\cdot$.
3. $\mathbb{Q}$, with the usual $+$ and $\cdot$.
4. $\mathbb{Z}$, with the usual $+$ and $\cdot$.
5. $\mathbb{R}$, with new operations $\oplus$ and $\otimes$ defined by $x \oplus y = x + y + 1$ and $x \otimes y = x \cdot y + x + y$ (where $+$ and $\cdot$ are the usual operations). Note that $x \oplus 0 \neq x$, but that doesn't mean it's not a field (there may be a different additive identity $Z$).
6. $\mathbb{C}$, with new operations $\oplus$ and $\otimes$ defined by $x \oplus y = x \cdot y$ and $x \otimes y = x + y$.

7. The set $\mathcal{P}(S)$ of all subsets of a given set $S$, with $\oplus$ the symmetric difference (i.e., the things in one set but not the other) and $\otimes$ the intersection.

8. The set $\{0, 1\}$, with new operations $\oplus$ and $\otimes$ defined as follows. The new sum $x \oplus y$ is 0 if $x + y$ is even and 1 if $x + y$ is odd, and the new product $x \otimes y$ is 0 if $xy$ is even and 1 if $xy$ is odd. In other words, add or multiply as usual and then take the remainder after dividing by 2.

9. The set $\{0, 1, 2\}$, where instead we take the remainder after dividing by 3.

10. The set $\{0, 1, 2, 3\}$, where instead we take the remainder after dividing by 4.

11. The set of positive real numbers, where $x \oplus y = x \cdot y$ and $x \otimes y = 10^{(\log_{10} x)(\log_{10} y)}$.

12. The set of all numbers of the form $x + yi$ with $x, y \in \mathbb{Q}$, with the usual $+$ and $\cdot$.

13. The set of all pairs $(x, y)$ with $x, y \in \mathbb{R}$, with the operations $(x, y) \oplus (w, z) = (x + w, y + z)$ and $(x, y) \otimes (w, z) = (xw, yz)$.