SIMUW Theory of Equations: Inequality Problems

1. Prove that
$$x + y + z \leq \sqrt{3x^2 + 3y^2 + 3z^2}.$$

2. For $x, y, z > 0$, prove that
$$\sqrt{\frac{x+y}{x+y+z}} + \sqrt{\frac{x+z}{x+y+z}} + \sqrt{\frac{y+z}{x+y+z}} \leq \sqrt{6}.$$

3. For $x, y, z > 0$, prove that
$$x + y + z \leq 2\left(\frac{x^2}{y+z} + \frac{y^2}{x+z} + \frac{z^2}{x+y}\right).$$

4. For $x, y > 0$ and integers $\alpha, \beta > 0$, prove that
$$x^\alpha y^\beta \leq \frac{\alpha}{\alpha+\beta} x^{\alpha+\beta} + \frac{\beta}{\alpha+\beta} y^{\alpha+\beta}.$$
In fact, deduce that it holds for all positive rationals α and β (from which it follows for all $\alpha, \beta > 0$ by continuity).

5. For $x, y > 0$, prove that
$$x^{2004}y + xy^{2004} \leq x^{2005} + y^{2005}.$$

6. For $x, y, z > 0$, prove that
$$3\sqrt[3]{xyz} \leq \frac{x^3}{yz} + \frac{y^3}{xz} + \frac{z^3}{xy}.$$

7. For $x, y, z > 0$, prove that
$$x + y + z \leq \frac{x^3}{yz} + \frac{y^3}{xz} + \frac{z^3}{xy}.$$
Note that by the A.M.-G.M. inequality, this is a stronger conclusion than in the previous problem, since $x + y + z \geq 3\sqrt[3]{xyz}$.

8. We saw in class that $(a^2+b^2)(x^2+y^2)-(ax+by)^2 = (ay-bx)^2$, which implies that it is nonnegative. The next case of Cauchy-Schwarz, namely
$$(a^2 + b^2 + c^2)(x^2 + y^2 + z^2) - (ax + by + cz)^2,$$
is not itself the square of a polynomial, but it is a sum of squares of polynomials. Prove that. Can you generalize?