

# SIMUW Theory of Equations: The surd plane

A surd is a real number that can be obtained (in finitely many steps) by starting with the rational numbers and then adding, subtracting, multiplying, dividing (by nonzero numbers, of course), and taking square roots (of positive numbers). For example,

$$\sqrt{1 + \sqrt{\frac{1 + \sqrt{5}}{2}} - \frac{2}{1 - \sqrt{7}} \sqrt{\sqrt{3}}}$$

is a surd.

The surd plane consists of the points in the plane whose coordinates are surds (these points are called surd points). A surd line is a line given by an equation

$$ax + by = c$$

where  $a, b, c$  are all surds (with  $a$  and  $b$  not both zero, so this equation defines a line). A surd circle is a circle given by an equation

$$(x - a)^2 + (y - b)^2 = r^2$$

where  $a, b, r$  are all surds.

Prove the following assertions:

1. The set of all surds forms a field.
2. A number  $x$  is a surd if and only if there is a sequence of fields  $F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq F_n \subseteq \mathbb{R}$  such that  $F_0 = \mathbb{Q}$ ,  $F_{j+1}$  is a quadratic extension of  $F_j$  for each  $j$ , and  $x \in F_n$ .
3. If two surd lines intersect, then the intersection point is a surd point.
4. If two surd circles intersect, then the intersection points are surd points.
5. If a surd line intersects a surd circle, then the intersection points are surd points.
6. Given two distinct surd points, the line through them is a surd line.
7. Given two distinct surd points  $P$  and  $Q$ , the circle centered at  $P$  that goes through  $Q$  is a surd circle.