

# SIMUW Theory of Equations

Here's a list of things you may need to know for this course. You are most likely already familiar with all of them from high school, but just in case it's best to find out now. If you're rusty, or have never studied some of these, please learn about them as soon as possible. Your fellow SIMUW students or the TACs can help you get up to speed.

- Completing the square; how to solve quadratic equations (including derivation of formula).
- How to solve simultaneous linear equations in many variables.
- What geometric and arithmetic series are and how to sum them.
- How to manipulate complex numbers (including complex conjugates).
- What radians are, and definitions of the trigonometric functions (but forget about the secant and cosecant).
- Simple trigonometric identities:  $\sin^2 \theta + \cos^2 \theta = 1$ , sine, cosine, or tangent of a sum, etc.
- Equations for lines and conic sections (particularly circles).
- Law of sines and law of cosines.

## Problems

When solving the problems in this class, always try to provide proofs, regardless of whether the problem explicitly asks for one. I encourage you to work together on the problems—the point is to get some experience in problem solving, not to compete with each other.

Please bring plenty of scratch paper to class. I typically won't leave any extra space on sheets of problems to work on during class. (You might need an arbitrarily large amount of space, so there's no point in even trying to accommodate it.)

1. Compute the square roots of  $i$ .
2. Factor  $x^4 + 4$  over  $\mathbb{C}$  and over  $\mathbb{Q}$ .
3. Show that  $\cos 2\pi/5 + i \sin 2\pi/5$  is a root of  $x^4 + x^3 + x^2 + x + 1$ .
4. Compute  $x^2 + 1/x^2$  in terms of  $x + 1/x$ .
5. Using the previous problem, solve  $x^4 + x^3 + x^2 + x + 1 = 0$  explicitly. (Hint: write it as  $x^2 + 1/x^2 + x + 1/x + 1 = 0$ .) Deduce formulas for  $\cos 2\pi/5$  and  $\sin 2\pi/5$ .
6. Show that a regular pentagon with side length 1 has diagonal length  $(1 + \sqrt{5})/2$ .
7. Prove that  $\sqrt{3}$  is irrational. Can you find an analogue for  $\sqrt{3}$  of each of the proofs we talked about for  $\sqrt{2}$ ? (Where do they break down for  $\sqrt{4}$ ?)

8. Prove that  $\sqrt{2} + \sqrt{3}$  is irrational. What about  $\sqrt{2} + \sqrt{3} + \sqrt{5}$ ?
9. A lattice point is a point in the plane with integer coordinates. Is there an equilateral triangle with its vertices at lattice points?
10. Find a formula for
- $$(\cos \theta)(\cos 2\theta)(\cos 4\theta) \dots (\cos 2^n \theta)$$
- (in closed form).

Exploration problem: Define  $a_0 = 1$  and  $a_n = \sin a_{n-1}$ . In other words,

$$a_n = \overbrace{\sin \dots \sin}^{n \text{ times}} 1.$$

What happens to  $a_n$  as  $n \rightarrow \infty$ ? It's not hard to see that it goes to 0, but how quickly? That problem is beyond the scope of this class, and we won't solve it. However, it's fun to explore and try to guess what happens, and proving it is a good challenge.