1. Given a weakly decreasing sequence \((a_0, a_1, \ldots)\) of natural numbers, let \(m = \min_{k \in \mathbb{N}} a_k\), and let \(n \in \mathbb{N}\) be the first natural number such that \(a_n = m\). Then \(a_k = m\) for all \(k \geq n\), because the sequence is weakly decreasing. Mapping \((a_0, a_1, \ldots)\) to \((a_0, a_1, \ldots, a_n)\) defines an injective function from the set of weakly decreasing sequences of natural numbers into \(\bigcup_{n \in \mathbb{N}} \mathbb{N}^{n+1}\). For each \(n\), \(\mathbb{N}^{n+1}\) is countable, and hence \(\bigcup_{n \in \mathbb{N}} \mathbb{N}^{n+1}\) is countable. Thus, the set of weakly decreasing sequences of natural numbers is countable.

2. There are uncountably many weakly increasing sequences of natural numbers. To show this, we will give an injective function from \(\{0, 1\}^\mathbb{N}\) to such sequences. Given \(x \in \{0, 1\}^\mathbb{N}\), define \((a_0, a_1, \ldots)\) by \(a_i = x_0 + x_1 + \cdots + x_i\). Then \((a_0, a_1, \ldots)\) is weakly increasing, and it determines \(x\) uniquely by \(x_0 = a_0\) and \(x_i = a_i - a_{i-1}\) for \(i > 0\).

3. The set \(\mathbb{Q}\) is countable and \(\mathbb{Q} \times A\) is countable for any countable \(A\), so we get that \(\mathbb{Q}^n\) is countable by induction. Note then that \(\bigcup_{n \in \mathbb{N}} \mathbb{Q}^n\) is a countable union of countable sets, and so is countable. Then mapping \((q_0, \ldots, q_n) \mapsto \{q_0, \ldots, q_n\}\) is a surjection from \(\bigcup_{n \in \mathbb{N}} \mathbb{Q}^n\) to the finite subsets of \(\mathbb{Q}\), and so there are only countably many finite subsets of \(\mathbb{Q}\).

4. Let \(s_0, s_1, \ldots\) be any sequence of rational numbers converging to \(\sqrt{2}\). (For example, it could be \(1, 1.4, 1.41, 1.414, \ldots\)) For each \(x \in \{-1, 1\}^\mathbb{N}\), define a new sequence \(s^x\) by \(s^x_i = s_i + x_i/(i+1)\). Then \(x\) is determined from \(s^x\) by \(x_i = (i+1)(s^x_i - s_i)\), so we have defined an injective mapping from \(x\) to \(s^x\). However, \(s^x\) is a rational sequence that again converges to \(\sqrt{2}\), because \(|s^x_i - s_i| = |x_i/(i+1)| = 1/(i+1) \to 0\) as \(i \to \infty\). Thus, \(-1, 1\}^\mathbb{N}\) is uncountable, there are uncountably many rational sequences converging to \(\sqrt{2}\).

2. (1) The function \(f : (0, 1) \to \mathbb{R}\) defined by \(f(x) = \tan(\pi(x - 1/2))\) is a bijection because it has the inverse function \(f^{-1}(x) = 1/2 + \arctan(x)/\pi\).

(2) Choose distinct elements \(x_0, x_1, x_2, \ldots\) of \((0, 1)\). Then we can define a bijection \(f : (0, 1) \to [0, 1]\) by \(f(x_0) = 0, f(x_1) = 1, f(x_{i+2}) = x_i\) for \(i \geq 0\), and \(f(y) = y\) for \(y \in (0, 1) \setminus \{x_0, x_1, \ldots\}\). This amounts to a bijection between the countably infinite sets \(\{x_0, x_1, \ldots\}\) and \(\{0, 1, x_0, x_1, \ldots\}\), while doing nothing to the rest of the interval.

(3) Because we can compose with a bijection between \(\mathbb{R}\) and \([0, 1]\), all we need to construct is a bijection between \([0, 1]\) and \(\{0, 1\}^\mathbb{N}\). Binary expansions almost work, but not quite, because two different binary expansions can give the same number. Specifically, there’s an ambiguity between those ending in all 0’s and in all 1’s. (E.g., \(1/2 = 0.1000\ldots = 0.0111\ldots\) in binary.) However, that’s the only ambiguity, and it occurs only for rational numbers whose denominators are powers of 2. We can fix it as follows.

Let \(A\) be the set of rational numbers in \([0, 1]\) whose denominators are powers of 2, and let \(B\) be the subset of \(\{0, 1\}^\mathbb{N}\) consisting of sequences that are eventually all 0’s or all 1’s. Then taking binary expansions gives a bijection between \([0, 1] \setminus A\) and \(\{0, 1\}^\mathbb{N} \setminus B\). Call it \(g : [0, 1] \setminus A \to \{0, 1\}^\mathbb{N} \setminus B\).

On the other hand, \(A\) and \(B\) are both countably infinite, \(A\) because the set of all rationals is countable, and \(B\) because for each \(k\) there are only finitely many sequences that are constant starting at the \(k\)-th term. Choose any bijection \(h : A \to B\) between \(A\) and \(B\), and then

\[
\begin{cases}
  g(x) & \text{if } x \in [0, 1] \setminus A, \\
  h(x) & \text{if } x \in A.
\end{cases}
\]

defines a bijection from \([0, 1]\) to \(\{0, 1\}^\mathbb{N}\).

(4) Using the bijection from (3), all we need is a bijection between \(\{0, 1\}^\mathbb{N}\) and \((\{0, 1\}^\mathbb{N})^2\). To do that, we just send a sequence \((a_0, a_1, \ldots) \in \{0, 1\}^\mathbb{N}\) to the pair \(((a_0, a_2, \ldots), (a_1, a_3, \ldots))\).

3. Every set \(S\) of disjoint open intervals in \(\mathbb{R}\) is countable. Specifically, define a function \(f : S \to \mathbb{Q}\) that maps each element of \(S\) to a rational number contained in it (there is a rational number in each open interval; the only reason why they have to be open is to avoid degenerate cases like \([\sqrt{2}, \sqrt{2}]\)). The disjointness of the
We will show that they are all countable, from which it follows that there must be an uncountable number of 
N4. There is an uncountable set of nested subsets of T. Suppose for all finite subsets

However, if S := \bigcup_{n,m} S_{n,m},

since each open interval (a, b) in S is contained in S_{n,m} when n > 1/(b - a) and m ≥ \max(|a|, |b|). Thus, S is a countable union of finite sets and hence countable.

4. There is an uncountable set of nested subsets of N. Because Q is in one-to-one correspondence with N, it suffices to find subsets of Q with the same property. For each r ∈ R, let Sr = \{x ∈ Q : x ≤ r\}. Then these sets are distinct subsets of Q, and they are nested because Sr1 ⊆ Sr2 whenever r1 ≤ r2. There are uncountably many of them because R is uncountable.

5. Again, there is an uncountable set. As above, it suffices to construct subsets of Q instead of N. For each real number r, pick a sequence s̃r, s̃r', ..., of rational numbers converging to r. Then the sets \{s̃i : i ∈ N\} are distinct for distinct r, because they have different limit points, and any two of these sets have finite intersection: if there were an infinite intersection, it would give a subsequence of both sequences, which would therefore converge to two different limits. Because R is uncountable, this gives an uncountable collection of subsets of Q with finite intersections.

Alternately, one can solve the problem as follows. Instead of considering subsets of N, we will consider subsets of \bigcup_{n≥1} \{0, 1\}^n; since both sets are countable, this is equivalent. Given r ∈ \{0, 1\}^N, let Sr = \{(r0, ..., rn) : n ≥ 1\}. Then (z0, ..., zn) ∈ Sr ∩ Ss only if

If r ≠ s, then there exists an i ∈ N such that ri ≠ si, and then Sr ∩ Ss has size at most i.

6. There exists an uncountable set of inequivalent sequences. Consider the equivalence classes of sequences. We will show that they are all countable, from which it follows that there must be an uncountable number of equivalence classes. (If there were only countably many, then there would be only countably many sequences in total.) Then taking one representative of each equivalence class gives an uncountable set of inequivalent sequences.

Thus, all we need to do is to show that the equivalence class of a sequence (a0, a1, ...) is countable. For each N, there are only countably many sequences (b0, b1, ...) such that a0 = b0 for all n ≥ N; the reason is that such a sequence differs from the original only in (b0, b1, ..., bN−1), which is an element of Q^N and thus has only countably many possibilities. Taking the union over all N gives a union of countably many countable sets, which remains countable.

7. Suppose for all finite subsets T ⊆ S, we have \sum_{s \in T} f(s) ≤ B. For each positive integer n, define Sn = \{s ∈ S : f(s) ≥ 1/n\}. Then for each n we must have |Sn| ≤ Bn, for otherwise we could take a finite subset T ⊆ Sn with |T| > Bn; then

contradicting our assumption. Then S' := \bigcup_{n∈N} Sn is a countable union of finite sets, and so is countable. However, if f(s) > 0 then f(s) ≥ 1/n for some positive integer n, so f(s) > 0 ⇒ s ∈ S', as desired.