RECOGNIZING CONIC SHAPE:
A NONLINEAR ITERATIVE APPROACH

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ABSTRACT

In this paper, a new algorithm for fitting a conic shape to a set of given data is described. By means of modified medial axis transform, described in the paper, the algorithm first determines the orientation of the shape such that the shape could be rotated to align its axis with the X-axis. The standard nonlinear equations of the conic, both closed and open, are then fitted using linearized least-square approach. Some experimental results are provided indicating the convergence and good performance of the algorithm.

## I. INTRODUCTION

Recognition and fitting of conic shapes have wide applications in image processing and computer vision [1]. A conic section is given by the equation $Q(x,y)=ax^2+bxy+cy^2+dx+ey+f=0$. The problem is to fit this curve treating $(x^2,xy,y^2,x,y,1)$ as data for a set of given $(x,y)$; where $(a,b,c,d,e,f)$ are the parameters to be estimated according to some minimum mean- square error criterion [2-5]. For a unique solution for the parameters $(a,b,c,d,e,f)$, a constraint equation is also required in the above mentioned formulation of the problem.

Bookstein [2] has elucidated the nature of mean-square error that is minimized in this general approach to fitting conic sections. His algorithm is designed to be 'invariant' with respect to equiform group of transformation of the Euclidean plane, i.e., rotations, translations and change of scale. Using the theory of invariants [6], the constraint equation is derived as $VDV'=2$ where $V=(a\ b\ c\ d\ e\ f)$ and D is the matrix diagonal (1 1/2 1 0 0 0). Moreover, Bookstein's algorithm can, also, incorporate linear constraints on the parameter values, i.e., $a=c$ and $b=0$ etc.

Bookstein's approach is still linear in form. In the present paper, the optimal curve fitting problem has been posed as a nonlinear parameter estimation problem transformed into a linearized least-square problem ; which could then be solved iteratively. The new algorithm is invariant with respect to translation and change of scale, but not with respect to rotation. A modified form of medial axis transform is used for the determination of the principal axis. Using this information, data scatter can be transformed into the standard form required by the algorithm. This axis transformation makes the algorithm rotation invariant.

## II. AXIS TRANSFORMATION

Consider the equation of an ellipse

$$\frac{(x-c)^2}{a^2} + \frac{(y-d)^2}{b^2} = 1 \tag{1}$$

where the set $(a,b,c,d)$ are the parameters of the ellipse. Equation (1) is valid only for ellipses with the major axis parallel to the X-axis. But, in general, this might not be the case. If the orientation $\varphi$ of the major axis is known, however, the original scatter $(x,y)$ could be transformed to a new scatter $(x,y)$ according to the relationship

$$x = x\cos\varphi - y\sin\varphi \quad ; \quad y = x\sin\varphi + y\cos\varphi \tag{2}$$

where the major axis of the new scatter is parallel to X-axis. To determine the orientation $\varphi$ , we propose modified medial axis transform ( MMAT ) to be discussed next.

**Modified Medial Axis Transform**

Consider the shape as given by Fig. 1. Let B be the set of points on the boundary of the shape. If for any interior point X of the shape, there exists two points on the boundary with equal minimum distance from X, then the point X is said to lie on the medial axis. The set of all X defines the medial axis. For circle, the medial axis is the center. For an ideal ellipse, or hyperbola or a parabola, the medial axis is a portion of the major axis. Thus, knowing the medial axis, the entire major axis could be known as the axis is a straight line by definition. Small changes in the shape, however, affects the medial axis very seriously [1]. To safeguard against such possibility, we modify the medial axis transform as follows.

To all points X belonging to medial axis, we assign a two element vector composed of the minimum distance of X from the boundary and the slope of the two minimum distant points on the boundary. This situation is depicted in Fig. 1(a). To find the modified medial axis, we first rank order the points belonging to the medial axis according to the first element of the vector ( $d$ parameter ). The bottom $\alpha$ % of the points belonging to medial axis are then discarded. This will ensure that the majority of the points which are only due to perturbation of the shape are eliminated. For the rest of the points on the medial axis, the second element ( the slope ) of the vector is histogrammed. The histogram is then searched for natural peaks using the segmentation algorithm described by Rosenfeld et al [7]. Usually, there is one well-defined peak for the points which define the major axis while the other points are due to perturbation and nonideal nature of the shape. For n peaks on the histogram, there are (n-1) thresholds which segments the slope range into n regions. Using these thresholds, the points on the medial axis are divided into n groups. A straight line is then fitted to all the points in a group according to least-square sense. That particular straight line would be considered the major axis which satisfy the following two conditions:

(1) The average square error ( square error per point ) is the least or comparable to it. The scheme for computation of the error has been discussed in the next section under the subheading "Stopping Criterion".

(2) The slope of the fitted straight line is perpendicular or nearly perpendicular to slope associated with most of the points in the group.

The straight line which satisfies the above mentioned conditions is called the modified medial axis and the process, the modified medial axis transform. The slope of this modified medial axis is then used in Eqn. (2) to determine a new scatter through which the standard equations of different conic sections could be fitted.

## III. LEAST SQUARE ESTIMATION

In this section, we pose the optimal curve fitting problem as an iterative linearized least-square problem. Towards this end, let us define

$$y = f(\ x,\ a_i,\ i=1,...,n\ ) \tag{3a}$$

An iterative scheme to determine $a_i$; $i=1,..,n$ could be written as

$$a_i^k = a_i^{k-1} + \nabla a_i \tag{3b}$$

where $k$ stands for the index of iteration. The initial values are given by ( $a_i^0, i=1,..,n$ ). Using Taylor's expansion of Eqn. (3a) and neglecting second and higher order derivatives, one can write

$$y = f(\, x, a_i^{k-1}+\nabla a_i, i=1,..,n\,) = f(\, x, a_i^{k-1}, i=1,..,n\,) +$$

$$\sum_i \frac{\partial f}{\partial a_i^{k-1}} \cdot \nabla a_i + \textit{higher order terms} \tag{4a}$$

Now, at a particular point ( $x_j, y_j$ ) at the $k$ th iteration, $f(\, x_j, a_i^{k-1}, i=1,..n)$ is known and is denoted by $y_j^e$. Substituting $y_j^e$ in Eqn. (4a) ,we could write

$$y_j - y_j^e = \sum_i \frac{\partial f}{\partial a_i^{k-1}} \cdot \nabla a_i + \textit{higher order terms} \tag{5}$$

for $j=1,..,m$. Equation (5) is in effect a set of $m$ equations where the partial derivatives are taken at points $(x_j, y_j)$. Consequently, equation (5) can be written in the matrix form as

$$\nabla y = A \nabla a + \underline{\Theta} \tag{6a}$$

where

$$\nabla y = \begin{bmatrix} y_1 - y_1^e \\ y_2 - y_2^e \\ \cdot \\ \cdot \\ y_m - y_m^e \end{bmatrix} \tag{6b}$$

$$\nabla a = \begin{bmatrix} \nabla a_1 \\ \cdot \\ \cdot \\ \nabla a_n \end{bmatrix} \tag{6c}$$

and

$$A = \begin{matrix} (\frac{\partial f}{\partial a_1^{k-1}})_1 & (\frac{\partial f}{\partial a_2^{k-1}})_1 & \cdot\ \cdot & (\frac{\partial f}{\partial a_n^{k-1}})_1 \\ (\frac{\partial f}{\partial a_1^{k-1}})_2 & (\frac{\partial f}{\partial a_2^{k-1}})_2 & \cdot\ \cdot & (\frac{\partial f}{\partial a_n^{k-1}})_2 \\ \cdot & \cdot & \cdot\ \cdot & \cdot \\ \cdot & \cdot & \cdot\ \cdot & \cdot \\ (\frac{\partial f}{\partial a_1^{k-1}})_m & (\frac{\partial f}{\partial a_2^{k-1}})_m & \cdot\ \cdot & -(\frac{\partial f}{\partial a_n^{k-1}})_m \end{matrix} \tag{6d}$$

$\underline{\Theta}$ is the error vector consisted of second and higher derivatives and measurement error of data; $(\frac{\partial f}{\partial a_i^{k-1}})_j$ stands for the partial derivative of $f(.)$ with respect to $a_i^{k-1}$ at the j-th point $(x_j, y_j)$ on the boundary . The minimum mean-square estimate of $\nabla a$ vector is given by [8]

$$\hat{\nabla} a = (A^T A)^{-1} A^T \begin{bmatrix} y_1 - y_1^e \\ \cdot \\ \cdot \\ y_m - y_m^e \end{bmatrix} \tag{7}$$

Once the estimate of $\nabla a$ is known , $a_i^k$ can be updated by means of Eqn. (3b). The iteration continues still there is no appreciable change in the values of $a_i$. The algorithm is next illustrated for the specific case of an ellipse. For the ellipse the parameter set ( $a,b,c,d$ ) is identified as $a_1=a$ , $a_2=b$ , $a_3=c$ and $a_4=d$; and

$$f(\, x, a,b,c,d\,) = y = d \pm \frac{b}{a}[a^2-(x-c)^2]^{1/2} \tag{8}$$

Knowing the functional form of $f(.)$, the partial derivatives could be calculated to construct the matrix of Eqn. (6d). Then, using the parameter values of the present iteration, $y_i^e$ could be calculated to determine the increment in the parameter values. In other words, let $\alpha$ , $\beta$ , $\gamma$ and $\delta$ be the corrections of $a,b,c$ and $d$ respectively. Then, the parameter values after $k$-th iteration is given by

$$a^k = a^{k-1} + \alpha \;;\; b^k = b^{k-1} + \beta \;;\; c^k = c^{k-1} + \gamma \;;\; d^k = d^{k-1} + \delta \tag{9}$$

**Stopping Criterion**

It is obvious from Eqn (7) that when $y_j$ is close to $y_j^e$ , $\nabla a$ is approximately a zero vector. As a result , $\alpha,\beta,\gamma,\delta$ which are the updates of $a,b,c$ and $d$ respectively , all tend to zero. Consequently, a distortion measure based on the difference $y_j - y_j^e$ for all values of $j$ is needed for the stopping criterion. A mean square difference function $D$ defined by

$$D = [\frac{1}{m} \sum_{j=1}^{j=m} (y_j - y_j^e)^2]^{1/2} \tag{10}$$

is next calculated. Under the tacit assumption that the domain of the independent variable of the given data scatter and the fitted curve are the same , the distortion measure given by Eqn. (10) is the one that is being minimized. For example, consider an ellipse with $m$ given points on the boundary ( $x_j, y_j, j=1,..,m$ ). We consider $x_j$ as the independent co-ordinate and $y_j$ as the dependent co-ordinate. Let the estimated parameter values be $\hat{a}, \hat{b}, \hat{c}$ and $\hat{d}$. We then calculate

$$y_j^e = \pm \frac{\hat{b}}{\hat{a}}[-(x_j-\hat{c})^2+\hat{a}^2]^{1/2} + \hat{d} \tag{11}$$

Two possible values of the left hand side of Eqn. (11) are due to the fact that an ellipse is a second - order curve. The value of $y_j^e$ which is closer to $y_j$ determines the proper sign. Observe that if the object boundary under consideration is a close approximation to an ellipse, each term of the summation in Eqn. (10) is small as $y_j$ is close to $y_j^e$. On the other hand, if the object is not an ellipse, most of the terms in the summation of Eqn. (10) tend to be large. As a result, if

$$\left| D \right| < \varepsilon \tag{12}$$

then the object under consideration is decided to be an ellipse with parameter values $\hat{a}, \hat{b}, \hat{c}$ and $\hat{d}$. Otherwise, the decision is negative. $\varepsilon$ is a parameter of the algorithm. A lower value of $\varepsilon$ demands greater conformity of the shape under consideration to an ellipse; and vice - versa.

## IV. INITIAL PARAMETER ESTIMATION

**CLOSED CONICS: { ELLIPSES and CIRCLES }**

Let $(X_1, Y_1)$ and $(X_2, Y_2)$ be the points of minimum and maximum distance, respectively, on the given shape from the origin. Then, it can be shown that for an ellipse in standard form

$$\alpha = \frac{Y_1 Y_2 (X_1 - X_2)}{X_1 Y_2 - X_2 Y_1} \;;\; \beta = \frac{X_1 X_2 (Y_1 - Y_2)}{X_1 Y_2 - X_2 Y_1} \tag{13a}$$

$$r^2 = \frac{Y_2^2 - Y_1^2 - 2\alpha(Y_1 - Y_2) + 2\beta(X_1 - X_2)}{X_1^2 - X_2^2 + 2\alpha(Y_1 - Y_2) + 2\beta(X_1 - X_2)} \tag{13b}$$

$$c = \frac{\beta(1-r^2)}{r^2} \;;\; d = \alpha(1-r^2) \tag{13c}$$

where, $b = ar$. The solution exists for $c$, $d$ and $r$ where $0 \le r \le 1$, only when the denominator of Eqn. (13a), i.e, $(X_1 Y_2 - X_2 Y_1)$, does not vanish. Also from Eqn. (1)

$$a^2 = (X_1 - c)^2 + \frac{(Y_1 - d)^2}{r^2} = (X_2 - c)^2 + \frac{(Y_2 - d)^2}{r^2} \tag{14}$$

Since $r$ is known, $a$ and, hence, $b$ is known from Eqn. (14).

{ **Special Case 1** } When $(X_1Y_2-X_2Y_1)=0$ , the object could be a circle with

$$c = \frac{X_1+X_2}{2};\ d = \frac{Y_1+Y_2}{2};\ 4r^2 = (X_1-X_2)^2+(Y_1-Y_2)^2 \quad (15)$$

where $(c,d)$ is the center of the circle and $r$ is the radius.

{ **Special Case 2**} The object could be an Ellipse with both $(X_1,Y_1)$ and $(X_2,Y_2)$ on either the major axis or the minor axis. If they are on the major axis,

$$c = \frac{X_1+X_2}{2};\ d = \frac{Y_1+Y_2}{2};\ 4a^2 = (X_1-X_2)^2+(Y_1-Y_2)^2 \quad (16)$$

knowing $a$,$c$ and $d$, $b$ can be calculated from Eqn. (1). Similarly, when $(X_1,Y_1)$ and $(X_2,Y_2)$ are on the minor axis,

$$c = \frac{X_1+X_2}{2};\ d = \frac{Y_1+Y_2}{2};\ 4b^2 = (X_1-X_2)^2+(Y_1-Y_2)^2 \quad (17)$$

knowing $b$,$c$ and $d$; $a$ can be calculated as before.

**OPEN CONICS: { HYPERBOLA and PARABOLA }**

By using the modified medial axis transform, the principal axis of the scatter is first aligned with the X-axis. The equation of a hyperbola is then given by

$$\frac{(x-c)^2}{a^2} - \frac{(y-d)^2}{b^2} = 1 \qquad (18)$$

where $(c,d)$ is the center of the hyperbola , and $(a,b)$ are the semi major and minor axis of the hyperbola.

Setting $b=ar$, differentiating Eqn. (18) with respect to $x$ and rearranging the terms , we get

$$y\frac{\partial y}{\partial x} = d\frac{\partial y}{\partial x} + r^2x - r^2c \qquad (19)$$

Selecting m points ( $x_j,y_j,j=1,..,m$ ) and approximating $\frac{\partial y}{\partial x}$ numerically, ( $d\ r^2\ r^2c$ ) can be calculated by linear least-square technique. If $r$ is found to be close to zero, the eccentricity is close to unity and a parabola is considered to be a more likely fit. Knowing $d$,$c$, and $r$ ; $a$ can then be evaluated using the equation

$$a^2 = \sum_{j=1}^{j=m} \frac{(x_j-c)^2}{m} - \frac{(y_j-d)^2}{mr^2} \qquad (20)$$

Once $a$ and $r$ is known, $b$ is also known. Similar procedure is used for the open conic **parabola.**

## V. SIMULATION RESULTS AND CONCLUSIONS

Figure 2 shows a closed shape along with the fitted ellipse. The distortion measure $D$ is calculated as 13.18 after 2 iterations. Figure 3 shows an open shape along with a fitted parabola; $D$=4.22 after two iterations.

On the basis of our investigations , the following comments are in order:

(1) If more than one MMAT exist, the algorithm has to carried out for each MMAT. Naturally, the best fit will determine the fitted conic.

(2) The order of convergence of this algorithm is one as the second and higher order derivatives are ignored. This suggests slow convergence. With good initial estimates, however, more than two iterations are not necessary, in general.

(3) The new algorithm could give good results for sparse data; but the sparse data should be distributed over the entire shape and not locally.

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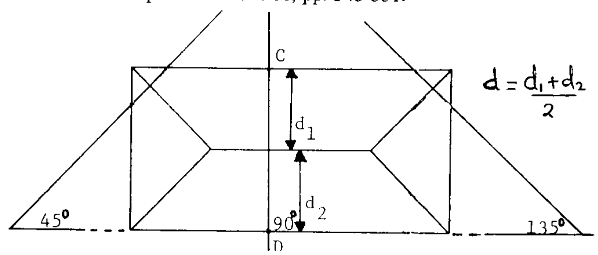


Figure 1

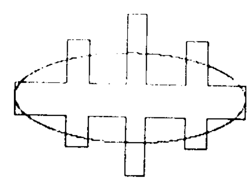

Figure 2

| INITIAL ESTIMATE | AFTER 2 ITERATIONS |
|---|---|
| A = 72.01 | A = 81.58 |
| B = 67.47 | B = 33.11 |
| C = 125.24 | C = 107.99 |
| D = 63.50 | D = 76.44 |
| DEVIATION = 32.67 | DEVIATION = 13.18 |

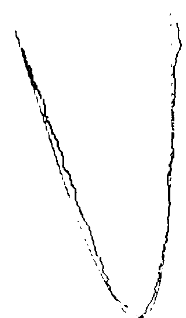

Figure 3