

What Makes Minimal Inference Tractable

Miki Hermann

joint work with

Arnaud Durand and Gustav Nordh

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Minimal Models

Reasoning with **minimal models** — widely used idea in

- artificial intelligence
- common sense and non-monotonic reasoning

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Main idea behind

- circumscription
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Algorithmic problems studied in connection with **minimal models**

- minimal model selection
- minimal model checking
- minimal inference (coincides with reasoning under ECWA)

Order and Minimality

Definition (Pointwise Partial Order, extending $0 < 1$)

Two Boolean vectors $m = (m[1], \dots, m[k])$ and $m' = (m'[1], \dots, m'[k])$ satisfy

$$m \leq m'$$

if $m[i] \leq m'[i]$ holds for each i . We write $m < m'$ for $m \leq m'$ and $m \neq m'$.

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Example

We have $010 < 011$, but 010 and 101 are incomparable.

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Definition (Minimal Model)

A satisfying assignment m is a **minimal model** of φ if there is no model m' of φ such that $m' < m$.

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Input: Two formulas φ and ψ .

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Remarks

- The **query** ψ sufficient to be a clause.
- **Π_2P -complete** in general, even for a literal ψ (Eiter and Gottlob, 1993).
- **coNP-complete** when φ is bijunctive or dual Horn (Cadoli and Lenzerini, 1994), or affine (Durand and H., 2003)
- In **P** when φ is Horn (common folklore).
- Classical inference $\varphi \models \psi$ is coNP-complete with subcases in P.

Special Boolean Relations

A Boolean relation R is

- **1-valid** if $1 \cdots 1 \in R$
- **0-valid** if $0 \cdots 0 \in R$
- **Horn** if R can be represented by a Horn formula, i.e. a CNF formula with at most one positive literal per clause
- **dual Horn** if R is represented by a dual Horn formula, i.e. a CNF formula with at most one negative literal per clause
- **bijunctive** if R is represented by a CNF formula with at most two literals per clause
- **affine** if R is represented by an affine system of equations $Ax = b$ over \mathbb{Z}_2
- **Schaefer** if it is Horn, dual Horn, bijunctive, or affine
- **complementive** if for each $(m_1, \dots, m_k) \in R$ also $(\neg m_1, \dots, \neg m_k) \in R$
- **incomparable** if *no* two vectors $m, m' \in R$ satisfy $m \leq m'$

Definition (Constraints)

- An atomic constraint $R(\vec{x})$ is an application of a Boolean relation R on a variable vector \vec{x}
- If $R_1(\vec{x})$ and $R_2(\vec{y})$ are constraints then $R_1(\vec{x}) \wedge R_2(\vec{y})$ is a constraint
- If $R(x_1, x_2, \vec{y})$ is a constraint then $R(x, x, \vec{y})$ is a constraint

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Notation

- $\langle S \rangle$ — the set of constraints/relations constructed from relations S by **conjunction** and **variable identification**
- $[\varphi]$ — the set of models of a constraint φ
- φ **represents** the relation $[\varphi]$

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Conjunction and variable identification only!

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Boolean Relations and Constraints

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Let $R = \{010, 101\}$. Both models 010 and 101 are minimal for $R(x, y, z)$, but there is only one minimal model for $\exists y R(x, y, z)$.

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Consequence

- The algebraic approach using Galois correspondence cannot be used
- $\text{Inv Pol } S \neq \langle S \rangle$
- Every proof must be done through explicit construction by means of conjunction and variable identification

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Remarks

- Kirousis and Kolaitis (2001, 2004) showed a **dichotomy** between the $\Pi_2\text{P}$ -complete cases from those included in coNP .
- They conjectured a **trichotomy** between $\Pi_2\text{P}$ -complete cases, coNP -complete cases, and cases in P .

Definition (0-section)

A relation R' is a **direct 0-section** of another relation $R \subseteq \{0, 1\}^k$ if there is an index i , such that

$$R' = \{(m[1], \dots, m[i-1], m[i+1], \dots, m[k]) \mid m \in R \text{ and } m[i] = 0\}$$

A relation R'' is a **0-section** of R if there is a sequence R_0, \dots, R_n , such that $R_0 = R$, $R'' = R_n$, and R_{j+1} is a direct 0-section of R_j .

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Definition (1-valid restriction)

The set S^* is a **1-valid restriction** of a set of relations S if it contains all relations R^* which are both 1-valid and 0-sections of a relation from S .

Proposition (Sharpening of a result from (Kirosis and Kolaitis, 2001))

Let S be a *non-Schaefer* and *non-0-valid* set of Boolean relations, with S^* being the corresponding *1-valid restriction*. If S^* is *Schaefer*, then $\text{MININF}(S)$ is coNP -complete, otherwise it is $\Pi_2\text{P}$ -complete.

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Proof.

For incomparable relations, minimal inference becomes classic inference $\varphi \models \psi$. Classic inference with φ Schaefer is in P . □

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Proof.

- A 0-valid constraint φ has the unique minimal model $0 \dots 0$.
- A Horn constraint φ , provided it is satisfiable, has a unique minimal model, which can be computed in polynomial time.



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Bijunctive Relations

Two special bijunctive relations: *nand* and *xor*

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Proposition

Let $N = \{[\neg x \vee \neg y], [x \neq y]\}$. $MININF(S)$ for each set of relations S , which is **bijunctive**, but neither other Schaefer, nor a subset of $\langle N \rangle$, is **coNP-complete**.

Theorem (Trichotomy of Minimal Inference)

Let S be a finite nonempty set of Boolean relations and S^* the corresponding 1-valid restriction. If every relation in S is

- Horn, or
- 0-valid, or
- both Schaefer and incomparable, or
- S can be constructed from *nand* and *xor* by conjunction and variable identification, i.e. S is a subset of $\langle N \rangle$, where $N = \{[\neg x \vee \neg y], [x \neq y]\}$,

then $\text{MININF}(S)$ is decidable in **polynomial time**. Else if S^* is Schaefer then $\text{MININF}(S)$ is **coNP-complete**. Otherwise $\text{MININF}(S)$ is **$\Pi_2\text{P}$ -complete**.

Checking conditions of a relation R in the Trichotomy Theorem

- if R is Schaefer — in polynomial time by closure properties
- if R is 0-valid or incomparable — in polynomial time by inspection
- if $S \subseteq \langle N \rangle$ — closure under majority plus Baker-Pixley Theorem

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- There are coNP-complete cases of MINIMAL INFERENCE for **dual Horn** φ with single literal queries (Cadoli and Lenzerini, 1994), but there exist tractable bounded query **dual Horn** MINIMAL INFERENCE problems.

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- All our results are for **unbounded** queries ψ .
- Our coNP-hardness proofs for the **bijunctive**, **affine**, and **dual Horn** cases are **not valid** for **bounded** queries.

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$\text{MININF}(S)$ with *bounded* queries for *bijunctive* relations S is in P.

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Open question

What is the complexity classification for $\text{MININF}(S)$ with *bounded* queries for S *affine* or *dual Horn*?

Concluding Remarks

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- The difficulty resides in proving all other cases to be **coNP**-hard.
- The complexity classification of $\text{MININF}(S)$ with **bounded** queries and **affine** S seems to be related to some representable matroid enumeration problems, whose complexity is widely open (Khachiyan, Boros, Elbassoni, Gurvich, and Makino, 2005).

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- The “simpler” problem restricted to **graphs** can be reduced to the t -DISJOINT PATH PROBLEM with bounded t , showed to be in P by a 300 page proof (Robertson and Seymour, 1995).

Thank you for your attention